JEE MAIN JAN 2020

PAPER WITH SOLUTION

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MATHEMATICS

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1. If $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then :

यदि $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$ है, तो :

(1) $\frac{1}{9} < I^2 < \frac{1}{8}$ (2) $\frac{1}{6} < I^2 < \frac{1}{2}$ (3) $\frac{1}{16} < I^2 < \frac{1}{9}$ (4) $\frac{1}{8} < I^2 < \frac{1}{4}$

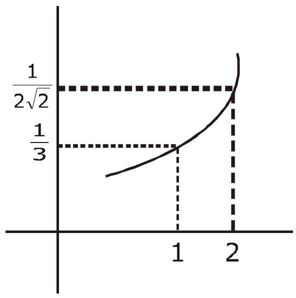
Sol. **1**

$I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$

$f(x) = \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$

$f'(x) = \frac{-1}{2}\frac{\left(6x^2 - 18x + 12\right)}{\left(2x^3 - 9x^2 + 12x + x\right)^{3/2}}$

$f'(x) = \frac{-6(x-1)(x-2)}{2\left(2x^3 - 9x^2 + 12x + 4\right)^{3/2}}$



$f(1) = \frac{1}{3}$ & $f(2) = \frac{1}{\sqrt{8}}$

it is increasing function

$\therefore \frac{1}{3} < I < \frac{1}{\sqrt{8}}$

$\frac{1}{9} < I^2 < \frac{1}{8}$

2. If a line, $y = mx + c$ is a tangent to the circle, $(x - 3)^2 + y^2 = 1$ and it is perpendicular to a line L_1, where L_1 is the tangent to the circle, $x^2 + y^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; then :

यदि एक रेखा $y = mx + c$, वत्त $(x - 3)^2 + y^2 = 1$ की एक स्पर्श रेखा है तथा यह एक रेखा L_1 पर लम्ब है, जहां L_1 वत्त $x^2 + y^2 = 1$ के बिन्दु $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ पर स्पर्श रेखा है, तो :

(1) $c^2 - 7c + 6 = 0$ (2) $c^2 + 7c + 6 = 0$ (3) $c^2 - 6c + 7 = 0$ (4) $c^2 + 6c + 7 = 0$

Sol. **4**

$(x - 3)^2 + y^2 = 1$, tangent is $y = mx + c$

for circle $x^2 + y^2 = 1$ tangnet at $P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

from T = 0

$$\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y-1=0$$

$x + y - \sqrt{2} = 0$ ......L_1

$x + y - \sqrt{2} = 0 \xrightarrow{\perp^r} x - y + \lambda = 0 \rightarrow$ This is tangent to the circle $(x - 3)^2 + y^2 = 1$

apply $r = \rho$

$$1 = \left|\frac{3-0+\lambda}{\sqrt{2}}\right| \Rightarrow |\lambda + 3| = \sqrt{2}$$

$\lambda = -3 \pm \sqrt{2}$

$(\lambda + 3)^2 = 2$

$\lambda^2 + 9 + 6\lambda - 2 = 0$

$c^2 + 6c + 7 = 0$

3. Which of the following statements is a tautology ?

निम्न में से कौन सा कथन एक पुनरूक्ति है ?

(1) $p \vee (\sim q) \rightarrow p \wedge q$ (2) $\sim (p \vee \sim q) \rightarrow p \vee q$

(3) $\sim (p \vee \sim q) \rightarrow p \wedge q$ (4) $\sim (p \wedge \sim q) \rightarrow p \vee q$

Sol. **2**

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \wedge q$	$A \rightarrow B$	$\sim (p \vee \sim q)$	$p \vee q$	$\sim (p \vee \sim q) \rightarrow (p \vee q)$
T	T	F	F	T	T	T	F	T	T
T	F	F	T	T	F	F	F	T	T
F	T	T	F	F	F	T	T	T	T
F	F	T	T	T	F	F	F	F	T

Tautolgy

4. Let $\alpha = \frac{-1+i\sqrt{3}}{2}$. If

$a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b = \sum_{k=0}^{100}\alpha^{3k}$, then

a and b are the roots of the quadratic equation :

माना $\alpha = \frac{-1+i\sqrt{3}}{2}$ है। यदि

$a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ तथा $b = \sum_{k=0}^{100}\alpha^{3k}$, तो

a तथा b निम्न में से किस द्विघात समीकरण के मूल हैं ?

(1) $x^2 - 102x + 101 = 0$ (2) $x^2 - 101x + 100 = 0$

(3) $x^2 + 102x + 101 = 0$ (4) $x^2 + 101x + 100 = 0$

Sol. 1

$\alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow \alpha = \omega$

$a = (1+\alpha)\sum_{k=0}^{100} \alpha^{2k}$

$a = (1 + \alpha)(1 + \alpha^2 + \alpha^4 + \alpha^6 + ...\alpha^{200})$

$a = (1+\alpha)\left\{\frac{1.\left[\left(\alpha^2\right)^{101} - 1\right]}{\alpha^2 - 1}\right\} \Rightarrow a = \frac{(\omega+1)\left(\omega^{202}-1\right)}{\left(\omega^2-1\right)}$

$\Rightarrow a = \frac{(\omega+1)(\omega-1)}{(\omega+1)(\omega-1)} \Rightarrow a = 1$

$b = \sum_{k=0}^{100} \alpha^{3k} = 1 + \alpha^3 + \alpha^6 + ...\alpha^{300}$

$b = 1 + \omega^3 + \omega^6 + ... \omega^{300} \Rightarrow b = 1 + 1 + +....101$ times

$b = 101$

$x^2 - (a + b)x + (ab) = 0$

$x^2 - (102)x + 101 = 0$

5. let $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ be two vectors. If $\vec{c}$ is a vector such that $\vec{b}\times\vec{c} = \vec{b}\times\vec{a}$ and $\vec{c}.\vec{a} = 0$, then $\vec{c}.\vec{b}$ is equal to :

माना दो सदिश $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ तथा $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ है। यदि एक सदिश $\vec{c}$ इस प्रकार है कि $\vec{b}\times\vec{c} = \vec{b}\times\vec{a}$ तथा $\vec{c}.\vec{a} = 0$ हैं, तो $\vec{c}.\vec{b}$ बराबर है :

(1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) $-\frac{3}{2}$ (4) -1

Sol. 2

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - \hat{j} + \hat{k}$

$|\vec{a}| = \sqrt{6}$; $|\vec{b}| = \sqrt{3}$ & $\vec{a}.\vec{b} = 4$

$\vec{b}\times\vec{c} = \vec{b}\times\vec{a}$

$(\vec{b}\times\vec{c}) - (\vec{b}\times\vec{a}) = 0$

$\vec{b}\times(\vec{c}-\vec{a}) = 0$

$\vec{b} \,||\, (\vec{c}-\vec{a}) \Rightarrow (\vec{c}-\vec{a}) = \lambda\vec{b}$

$\vec{c} = \vec{a} + \lambda\vec{b}$

Now $\vec{c}.\vec{a} = 0$

$\Rightarrow \vec{c}.\vec{a} = \vec{a}.\vec{a} + \lambda\left(\vec{a}.\vec{b}\right)$

$0 = |\vec{a}|^2 + \lambda\left(\vec{a}.\vec{b}\right)$

$\lambda = \dfrac{-|\vec{a}|^2}{\vec{a}.\vec{b}} = \dfrac{-6}{4} = \dfrac{-3}{2}$

$\therefore\ \vec{c} = \vec{a} - \dfrac{3}{2}\vec{b}$

$\vec{c} = \left(\hat{i} - 2\hat{j} + \hat{k}\right) - \dfrac{3}{2}\left(\hat{i} - \hat{j} + \hat{k}\right)$

$\vec{c} = -\dfrac{1}{2}\hat{i} - \dfrac{1}{2}\hat{j} - \dfrac{1}{2}\hat{k} \Rightarrow C = -\dfrac{1}{2}\left(\hat{i} + \hat{j} + \hat{k}\right)$

$\vec{b}.\vec{c} = -\dfrac{1}{2}\left(\hat{i} + \hat{j} + \hat{k}\right)\left(\hat{i} - \hat{j} + \hat{k}\right)$

$\vec{b}.\vec{c} = -\dfrac{1}{2}$

6. If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $\left(x + \sqrt{x^2 - 1}\right)^6 + \left(x - \sqrt{x^2 - 1}\right)^6$, then :

यदि $\left(x + \sqrt{x^2 - 1}\right)^6 + \left(x - \sqrt{x^2 - 1}\right)^6$ के प्रसार में x^4 तथा x^2 के गुणांक क्रमशः α तथा β हैं, तो :

(1) $\alpha - \beta = 60$ (2) $\alpha - \beta = -132$ (3) $\alpha + \beta = 60$ (4) $\alpha + \beta = -30$

Sol. **2**

$(x+a)^n + (x - a)^n = 2(T_1 + T_3 + T_5 + ...)$

$\left(x + \sqrt{x^2 - 1}\right)^6 + \left(x - \sqrt{x^2} - 1\right)^6 = 2[T_1 + T_3 + T_5 + T_7]$

$= 2[{}^6C_0\, x^6 + {}^6C_2 x^4(x^2 - 1) + {}^6C_4 x^2(x^2 - 1)^2 + {}^6C_6 x^0(x^2 - 1)^3]$

$= 2[x^6 + 15(x^6 - x^4) + 15x^2\,(x^4 + 1 - 2x^2) + [x^6 - 3x^4 + 3x^2 - 1)]$

$= 2[x^6(2 + 15 + 15 + 1) + x^4(-15 - 30 - 3) + x^2(15+3)]$

coefficient of $x^4 = \alpha = -96$

$\beta = 36$

$\alpha - \beta = -96 - 36 = -132$

7. If a hyperbola passes through the point P(10,16) and it has vertices at (±6,0), then the equation of the normal to it at P is :

यदि एक अतिपरवलय बिन्दु P(10, 16) से होकर जाता है तथा इसके शीर्ष (±6,0) पर हैं, तो P पर इसके अभिलम्ब का समीकरण है :

(1) x + 2y = 42 (2) x + 3y = 58 (3) 2x + 5y = 100 (4) 3x + 4y = 94

Sol. **3**

Let hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

vertices (±a, 0) = (±6, 0) ⇒ a = 6

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \xrightarrow{(10,16)} \frac{(10)^2}{(6)^2}-\frac{(16)^2}{b^2} = 1 \Rightarrow \frac{256}{b^2} = \frac{100}{36}-1$$

$$\frac{256}{b^2} = \frac{64}{36}$$

$$b^2 = \frac{256\times 36}{64} \Rightarrow b^2 = 36\times 4$$

$b^2 = 9\times 16$

$b = 12$

∴ required hyperbola is $\frac{x^2}{36}-\frac{y^2}{144}=1$

equation of normal will be

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2 + b^2$$

at P(10,16) normal is

$$\frac{36x}{10}+\frac{144y}{16} = 36 + 144$$

$$\frac{18x}{5}+9y = 180$$

$18x + 45y = 900$

$2x + 5y = 100$

8. $\lim\limits_{x\to 0}\frac{\int_0^x t\sin(10t)\,dt}{x}$ is equal to :

$\lim\limits_{x\to 0}\frac{\int_0^x t\sin(10t)\,dt}{x}$ बराबर है :

(1) $\frac{1}{10}$ (2) $-\frac{1}{5}$ (3) $\frac{-1}{10}$ (4) 0

Sol. **4**

$$\lim_{x\to 0}\frac{\int_0^x t\sin(10t)\,dt}{x}$$

$\frac{0}{0}$ form

$\therefore$ apply newton leibniz rule

$$\lim_{x\to 0}\frac{x.\sin(10x)-0}{1} = 0$$

9. The system of linear equations

$\lambda x + 2y + 2z = 5$

$2\lambda x + 3y + 5z = 8$

$4x + \lambda y + 6z = 10$ has :

(1) no solution when $\lambda = 2$

(2) infinitely many solutions when $\lambda = 2$

(3) no solution when $\lambda = 8$

(4) a unique solution when $\lambda = -8$

रैखिक समीकरण निकाय

$\lambda x + 2y + 2z = 5$

$2\lambda x + 3y + 5z = 8$

$4x + \lambda y + 6z = 10$

(1) का कोई हल नही है जब $\lambda = 2$

(2) के अनन्त हल हैं जब $\lambda = 2$

(3) का कोई हल नहीं है जब $\lambda = 8$

(4) का मात्र एक हल है जब $\lambda = -8$

Sol. **1**

$\lambda x + 2y + 2z = 5$

$2\lambda x + 3y + 5z = 8$

$4x + \lambda y + 6z = 10$

$$\Delta = \begin{vmatrix} \lambda & 2 & 2 \\ 2\lambda & 3 & 5 \\ 4 & \lambda & 6 \end{vmatrix}$$

$\Delta = \lambda(18 - 5\lambda) - 2(12\lambda - 20) + 2(2\lambda^2 - 12)$

$\Delta = 18\lambda - 5\lambda^2 - 24\lambda + 40 + 4\lambda^2 - 24$

$\Delta = -\lambda^2 - 6\lambda + 16$

$\Delta = -\lambda^2 + 2\lambda - 8\lambda + 16$

$\Delta = -\lambda(\lambda - 2) - 8(\lambda - 2)$

$\Delta = -(\lambda + 8)(\lambda - 2)$

for no solutions $\Delta = 0 \Rightarrow \lambda = -8, \lambda = 2$

when $\lambda = 2$

$$\Delta_1 = \begin{vmatrix} 2 & 2 & 2 \\ 4 & 3 & 5 \\ 4 & 2 & 6 \end{vmatrix} = 2(18-10) - 2(24-20) + 2(8-12)$$

$= 2(2) - 2(4) + 2(-4)$

$\Delta_1 \neq 0$

$\therefore$ at $\lambda = 2$ no solutions

10. If the 10th term of an A.P. is $\frac{1}{20}$ and its 20th term is $\frac{1}{10}$, then the sum of its first 200 terms is :

यदि एक समान्तर श्रेढ़ी का 10 वां पद $\frac{1}{20}$ है तथा इसका 20 वां पद $\frac{1}{10}$ है, तो इसके प्रथम 200 पदों का योग है :

(1) 100 (2) $50\frac{1}{4}$ (3) 50 (4) $100\frac{1}{2}$

Sol. **4**

$T_{10} = a + 9d = \frac{1}{20}$ ...(1)

$T_{20} = a + 19d = \frac{1}{10}$ ....(2)

Equation (2) – (1)

$-10d = \frac{1}{20} - \frac{1}{10}$

$-10d = \frac{1}{20} - \frac{2}{20} = \frac{-1}{20}$

$d = \frac{1}{200}$

$a + \frac{9}{200} = \frac{1}{20} \Rightarrow a = \frac{1}{20} - \frac{9}{200}$

$a = \frac{10}{200} - \frac{9}{200} \Rightarrow a = \frac{1}{200}$

$\therefore a = d = \frac{1}{200}$

$S_{200} = \frac{200}{2}\left[\frac{2}{200} + (200-1).\frac{1}{200}\right] = 100\left[\frac{2}{200} + \frac{199}{200}\right]$

$S_{200} = \frac{201}{2} = 100\frac{1}{2}$

11. Let f : (1,3) $\to$ R be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where [x] denotes the greatest integer $\le$ x. Then the range of f is :

माना f : (1,3) $\to$ R एक फलन है, जो $f(x) = \frac{x[x]}{1+x^2}$, द्वारा परिभाषित है जहां [x] महत्तम पूर्णांक $\le$ x को दर्शाता है। तो f का परिसर है :

(1) $\left(\frac{2}{5},\frac{1}{2}\right) \cup \left(\frac{3}{5},\frac{4}{5}\right]$ (2) $\left(\frac{2}{5},\frac{4}{5}\right]$ (3) $\left(\frac{3}{5},\frac{4}{5}\right)$ (4) $\left(\frac{2}{5},\frac{3}{5}\right] \cup \left(\frac{3}{4},\frac{4}{5}\right)$

Sol. **1**

$f:(1,3) \to R$, $f(x) = \frac{x[x]}{1+x^2}$

$$f(x) = \begin{cases} \frac{x}{1+x^2}, & x \in (1,2) \\ \frac{2x}{1+x^2}, & x \in [2,3) \end{cases}$$

$$f'(x) = \begin{cases} \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}, & x \in (1,2) \\ \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2}, & x \in [2,3) \end{cases}$$

$$f'(x) = \begin{cases} \frac{1-x^2}{1+x^2}, & x \in (1,2) \\ \frac{1-2x^2}{1+x^2}, & x \in [2,3) \end{cases}$$

$\therefore$ f(x) is decreasing function

$\therefore R_f \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{6}{10}, \frac{4}{5}\right]$

12. The differential equation of the family of curves, $x^2 = 4b(y + b)$, $b \in R$, is :

वक्रों $x^2 = 4b(y + b)$, $b \in R$ के कुल का अवकल समीकरण है :

(1) $x(y')^2 = 2yy' - x$ (2) $xy'' = y'$ (3) $x(y')^2 = x - 2yy'$ (4) $x(y')^2 = x + 2yy'$

Sol. **4**

$x^2 = 4b(y + b)$, $b \in R$

$x^2 = 4by + 4b^2$

$2x = 4by' + 0$

$$y' = \frac{x}{2b}$$

$$\Rightarrow b = \frac{x}{2y'}$$

$\therefore$ differential equation is

$$x^2 = 4.y.\frac{x}{2y'} + 4\left(\frac{x}{2y'}\right)^2$$

$$x^2 = \frac{2xy}{y'} + \frac{x^2}{(y')^2}$$

$$x = \frac{2y}{y'} + \frac{x}{(y')^2}$$

$x(y')^2 = 2yy' + x$

13. Let S be the set of all functions $f:[0,1] \rightarrow R$, which are continuous on [0,1] and differentiable on (0,1) . Then for every f in S, there exists a $c \in (0,1)$, depending on f, such that :

माना सभी फलनों $f : [0,1] \rightarrow R$, जो कि [0,1] पर संतत हैं तथा (0,1) पर अवकलनीय हैं, का समुच्चय S है। तो S में प्रत्येक f के लिए f पर निर्भर एक $c \in (0,1)$ का अस्तित्व इस प्रकार है कि :

(1) $\frac{f(1)-f(c)}{1-c} = f'(c)$ (2) $|f(c) - f(1)| < (1 - c)\,|f'(c)|$

(3) $|f(c) - f(1)| < |f'(c)|$ (4) $|f(c) + f(1)| < (1+c)|f'(c)|$

Sol. 4

Use LMVT theorem & check option

14. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is :

20 प्रेक्षणों के माध्य तथा प्रसरण क्रमशः 10 तथा 4 पाये गये। पुनः जांच करने पर पाया गया कि एक प्रेक्षण 9 गलत था तथा सही प्रेक्षण 11 था। तो सही प्रसरण है :

(1) 4.01 (2) 4.02 (3) 3.98 (4) 3.99

Sol. 4

Let 20 observation be $x_1, x_2, \ldots x_{20}$

given Mean $= \frac{x_1 + x_2 + \ldots\ldots + x_{20}}{20} = 10$

$x_1 + x_2 + \ldots + x_{20} = 200$

Now, $x_1 + x_2 + \ldots + x_{20} - 9 + 11 \Rightarrow 202$

$V_{ar} = \frac{\sum x_i^2}{n} - (\bar{x})^2$

$4 = \frac{x_1^2 + x_2^2 + \ldots + x_{20}^2}{20} - 10^2$

$2080 = x_1^2 + x_2^2 + \ldots\ldots + x_{20}^2$

Now, $x_1^2 + x_2^2 + \ldots\ldots + x_{20}^2 - 81 + 121 \Rightarrow 2080 + 40 = 2120$

new variance will be

$\frac{2120}{20} - \left(\frac{202}{20}\right)^2 = 3.99$

15. The area (in sq. units) of the region $\{(x,y) \in R^2 : x^2 \le y \le 3 - 2x\}$, is :

क्षेत्र $\{(x,y) \in R^2 : x^2 \le y \le 3 - 2x\}$ का क्षेत्रफल (वर्ग इकाई में) है :

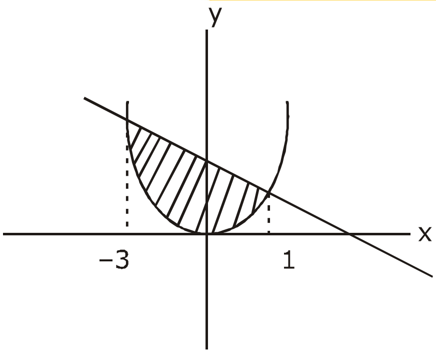
(1) $\frac{31}{3}$ (2) $\frac{34}{3}$ (3) $\frac{29}{3}$ (4) $\frac{32}{3}$

Sol. 4

$y = x^2$ & $y = 3 - 2x$

$x^2 = 3 - 2x$

$x^2 + 2x - 3 = 0$
$x^2 + 3x - x - 3 = 0$
$x(x + 3) - (x + 3) = 0$
$(x - 1)(x + 3) = 0$
$x = 1, -3$



required area = $\int_{-3}^{1} (\text{line} - \text{parabola})\,dx$

$= \int_{-3}^{1} \left[(3 - 2x) - x^2\right]$

$= \left(3x - \frac{2x^2}{2} - \frac{x^3}{3}\right)_{-3}^{1}$

Area = $\left(3x - x^2 - \frac{x^3}{3}\right)_{-3}^{1}$

$= \left[3(1) - (1)^2 - \frac{1}{3}\right] - \left[3(-3) - (-3)^2 - \frac{(-3)^3}{3}\right]$

$= \left(2 - \frac{1}{3}\right) - (-18 + 9)$

$= \frac{5}{3} + 9 = \frac{32}{3}$

16. The length of the perpendicular from the origin, on the normal to the curve, $x^2 + 2xy - 3y^2 = 0$ at the point (2,2) is

वक्र $x^2 + 2xy - 3y^2 = 0$ के बिन्दु (2, 2) पर खींचे गये अभिलम्ब पर मूल बिन्दु से डाले गये लम्ब की लम्बाई है :

(1) 2 (2) $\sqrt{2}$ (3) $2\sqrt{2}$ (4) $4\sqrt{2}$

Sol. **3**

$x^2 + 2xy - 3y^2 = 0$
$2x + 2y + 2xy' - 6yy' = 0$
$x + y + xy' - 3yy' = 0$
$y'(x - 3y) = -(x + y)$

$\frac{dy}{dx} = \frac{x + y}{3y - x}$

Slope of (N), $-\frac{dx}{dy} = \frac{x - 3y}{x + y}$

$\therefore \left(-\frac{dx}{dy}\right)_{(2,2)} = \frac{2 - 6}{2 + 2} = -1$

$\therefore$ equation of normal at (2,2) is

$y - 2 = -(x - 2)$

$x + y - 4 = 0$

$\therefore$ distance from (0,0) will be

$$p = \left|\frac{0+0-4}{\sqrt{2}}\right| = 2\sqrt{2}$$

17. Let S be the set of all real roots of the equation, $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$. Then S :

(1) is a singleton
(2) contains at least four elements.
(3) contains exactly two elements
(4) is an empty set.

माना समीकरण $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$ के सभी वास्तविक मूलों का समुच्चय S है। तो S :

(1) एक ही अवयव वाला समुच्चय है।
(2) में कम से कम चार अवयव हैं।
(3) में मात्र दो अवयव हैं
(4) एक रिक्त समुच्चय है

Sol. **1**

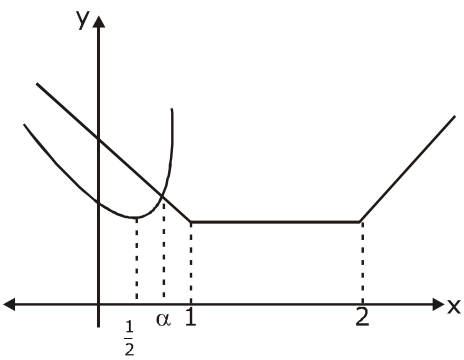
$3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$

put $3^x = t$

$t(t - 1) + 2 = |t - 1| + |t - 2|$

$t^2 - t + 2 = |t - 1| + |t - 2|$

from graph



let α is real solutions

$\alpha = 3^x$

$x = \log_3\alpha$ $\therefore$ only one solution

$\therefore$ singleton set

18. If $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $10A^{-1}$ is equal to :

यदि $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ तथा $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ हैं, तो $10A^{-1}$ बराबर है :

(1) A −6I
(2) 4I − A
(3) 6I −A
(4) A − 4I

Sol. **1**

$$A = \begin{bmatrix} 2 & 2 \\ 9 & 4 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$10A^{-1} = ?$
According to Cayley Hamilton equation
$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 \\ 9 & 4-\lambda \end{vmatrix} = 0$$

$(2 - \lambda)(4 - \lambda) - 18 = 0$
$8 - 2\lambda - 4\lambda + \lambda^2 - 18 = 0$
$\lambda^2 - 6\lambda - 10 = 0$
$\therefore A^2 - 6A - 10 = 0$
$A^{-1}(A^2) - 6A^{-1}A - 10A^{-1} = 0$
$A - 6I - 10A^{-1} = 0$
$10A^{-1} = A - 6I$

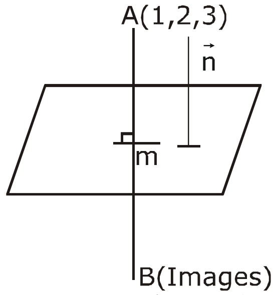
19. The mirror image of the point (1,2,3) in a plane is $\left(-\frac{7}{3},-\frac{4}{3},-\frac{1}{3}\right)$. Which of the following point lies on this plane ?

बिन्दु (1,2,3) का एक समतल में प्रतिबिम्ब (mirror image), $\left(-\frac{7}{3},-\frac{4}{3},-\frac{1}{3}\right)$ है। निम्न में से कौन सा बिन्दु इस समतल पर स्थित है ?

(1) (1,1,1) (2) (1,–1,1) (3) (–1,–1,–1) (4) (–1,–1,1)

Sol. **2**

for required plane



$\vec{n} \,||\, \overline{AB}$

$\vec{n} = -\frac{7}{3}-1, \frac{-4}{3}-2, \frac{-1}{3}-3$

$\vec{n} = \frac{-10}{3}, \frac{-10}{3}, \frac{-10}{3}$

D.r of $\vec{n}$ = 1,1,1

Also mid- point of A & B is $M \equiv \left(\frac{\frac{-7}{3}+1}{2}, \frac{\frac{-4}{3}+2}{2}, \frac{\frac{-1}{3}+3}{2}\right)$

$M \equiv \left(\frac{-4}{6}, \frac{2}{6}, \frac{8}{6}\right)$

$M \equiv \left(\frac{-2}{3}, \frac{1}{3}, \frac{4}{3}\right)$

$\therefore$ equation of required plane B

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right) = -\frac{2}{3}+\frac{1}{3}+\frac{4}{3}$

$x + y + z = 1$

20. Let A and B be two events such that the probability that exactly one of them occurs is $\frac{2}{5}$ and the probability that A or B occurs is $\frac{1}{2}$, then the probability of both of them occur together is :

माना A तथा B दो घटनाऐं इस प्रकार है कि दोनों में से मात्र एक के होने की प्रायिकता $\frac{2}{5}$ है तथा A या B के होने की प्रायिकता $\frac{1}{2}$ है, तो दोनो के एक साथ होने की प्रायिकता है :

(1) 0.02 (2) 0.01 (3) 0.20 (4) 0.10

Sol. **4**

P(exactly one) = $\frac{2}{5}$

$P(A) + P(B) - 2P(A \cap B) = \frac{2}{5}$ ....(1)

$P(A \text{ or } B) = P(A \cup B) = \frac{1}{2}$

$P(A) + P(B) - P(A \cap B) = \frac{1}{2}$ ....(2)

$P(A \cap B) = ?$

Solve (1) – (2)

$-P(A \cap B) = \frac{2}{5} - \frac{1}{2}$

$P(A \cap B) = \frac{1}{2} - \frac{2}{5}$

$= \frac{5}{10} - \frac{4}{10} = \frac{1}{10}$

$= P(A \cap B) = 0.10$

21. The number of 4 letters words (with or without meaning) that can be formed from the eleven letters of the word 'EXAMINATION' is ....

शब्द 'EXAMINATION' के ग्यारह अक्षरों से बन सकने वाले 4 अक्षरों के शब्दों (अर्थ वाले तथा अर्थविहीन) की संख्या है.............. ।

Sol. **2454**

EXAMINATION

(AA)(II)(NN)(EXMOT)

to form four letter words

(1) All same ....not possible

(2) 1 different, 3 same.............not possible

(3) 2 different, 2 same..........$^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 3 \times \frac{7 \times 6 \times 5!}{2!5!} \times \frac{4 \times 3 \times 2!}{2!} = 63 \times 12 = 756$

(4) 2 same, 2 same............$^3C_2 \times \frac{4!}{2!2!} = 3 \times \frac{4 \times 3 \times 2!}{2.2} = 18$

(5) All different ....$^8C_4 \times 4! = \dfrac{8.7.6.5.4!}{4.3.2.4!} \times 4! = 56 \times 30 = 1680$

Total = 2454

22. If $\dfrac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \dfrac{1}{7}$ and $\sqrt{\dfrac{1-\cos 2\beta}{2}} = \dfrac{1}{\sqrt{10}}$, $\alpha, \beta \in \left(0, \dfrac{\pi}{2}\right)$, then $\tan(\alpha + 2\beta)$ is equal to :

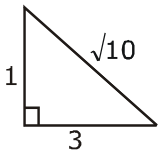
यदि $\dfrac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \dfrac{1}{7}$ तथा $\sqrt{\dfrac{1-\cos 2\beta}{2}} = \dfrac{1}{\sqrt{10}}$, $\alpha, \beta \in \left(0, \dfrac{\pi}{2}\right)$, हैं, तो $\tan(\alpha + 2\beta)$ बराबर है.......... ।

Sol. 1

$$\frac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \frac{1}{7} \text{ and } \sqrt{\frac{1-\cos 2\beta}{2}} = \frac{1}{\sqrt{10}}$$

$$\frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha} = \frac{1}{7} \ \& \ \frac{\sqrt{2}\sin\beta}{\sqrt{2}} = \frac{1}{\sqrt{10}}$$

$$\tan\alpha = \frac{1}{7} \qquad \sin\beta = \frac{1}{\sqrt{10}}$$



$$\tan\beta = \frac{1}{3}$$

$$\therefore \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\left(\frac{1}{3}\right)}{1-\frac{1}{9}} = \frac{2/3}{8/9}$$

$$\tan 2\beta = \frac{2}{3}\cdot\frac{9}{8} = \frac{3}{4}$$

$$= \tan(\alpha + 2\beta)$$

$$= \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha.\tan 2\beta}$$

$$= \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}} = \frac{\frac{4+21}{28}}{\frac{28-3}{28}} = \frac{25}{25} = 1$$

23. The sum $\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$ is equal to :

योगफल $\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$ बराबर है .... ।

Sol. **504**

$\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$

$\frac{1}{4}\sum_{n=1}^{7}n(n+1)(2n+1)$

$\frac{1}{4}\sum_{n=1}^{7}\left((n^2+n)(2n+1)\right)$

$= \frac{1}{4}\sum_{n=1}^{7}\left(2n^3+3n^2+n\right)$

$\frac{1}{2}\sum_{n=1}^{7}n^3 + \frac{3}{4}\sum_{n=1}^{7}n^2 + \frac{1}{4}\sum_{n=1}^{7}n$

$\Rightarrow \frac{1}{2}\left(\frac{7(7+1)}{2}\right)^2 + \frac{3}{4}\left(\frac{7(7+1)(14+1)}{6}\right) + \frac{1}{4}\frac{7(8)}{2}$

$= \frac{1}{2}\frac{49.8.8}{4} + \frac{3.7.8.15}{4.6} + \frac{1}{4}\frac{7.8}{2}$

$= (49)(8) + (15\times7) + (7)$

$= 392 + 105 + 7 = 504$

24. Let f(x) be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, f(x) has a critical point at $x = -1$ and f'(x) has a critical point at $x = 1$. Then f(x) has a local minima at x = ...

माना घात 3 का एक बहुपद f(x) इस प्रकार है कि $f(-1) = 10$, $f(1) = -6$, f(x) का एक क्रांतिक बिन्दु $x = -1$ है तथा f'(x) का एक क्रांतिक बिन्दु $x = 1$ है। तो f(x) का एक स्थानीय निम्ननिष्ठ है x = ...है।

Sol. **3**

$f(x) = ax^3 + bx^2 + cx + d$

$f(-1) = 10,\ f(1) = -6$

$-a + b - c + d = 10$ ....(i)

$a + b + c + d = -6$ ......(ii)

add (i) + (ii)

$2(b + d) = 4$

$b + d = 2$ ....(iii)

$f'(x) = 3ax^2 + 2bx + c$

$f'(-1) = 0$

$3a - 2b + c = 0$ .....(iv)

$f''(x) = 6ax + 2b$

$f''(1) = 0$

$6a + 2b = 0$ ....(v)

add (iv) + (v)

$9a + c = 0$ ....(vi)

$b = -3a$

$c + 9\left(\frac{-b}{3}\right) = 0$

$c = 3b$

$f(x) = \frac{-b}{3}x^3 + bx^2 + 3bx + (2-b)$

$f'(x) = -bx^2 + 2bx + 3b = 0$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3, -1$ Minima

at $x = 3$

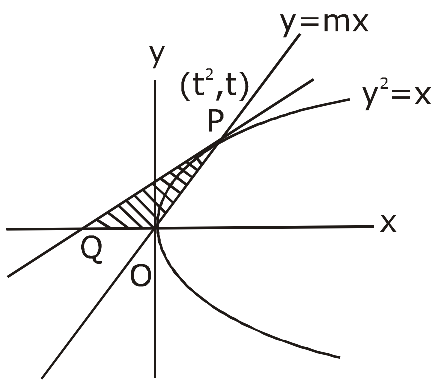
25. Let a line $y = mx (m > 0)$ intersect the parabola, $y^2 = x$ at a point P, other than the origin. Let the tangent to it at P meet the x - axis at the point Q. If area $(\Delta OPQ) = 4$ sq. units, then m is equal to ....

माना एक रेखा $y = mx (m > 0)$, परवलय $y^2 = x$ को मूल बिन्दु के अतिरिक्त एक बिन्दु P पर काटती है। माना P पर इसकी स्पर्श रेखा x-अक्ष को बिन्दु Q पर मिलती है। यदि (ΔOPQ) का क्षेत्रफल 4 वर्ग इकाई है, तो m बराबर है ....।

Sol. **0.5**

let $p(t^2, t)$

Tangent at $P(t^2, t)$



$ty = \frac{x + t^2}{2}$

$2ty = x + t^2 \rightarrow$ equation of tangent

$Q \equiv (-t^2, 0)$

$O(0,0)$

$\Delta(OPQ) = 4$

$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ t^2 & t & 1 \\ -t^2 & 0 & 1 \end{vmatrix} = 4$

$|t^3| = 8$

$t = 2$

$\therefore 4y = x + 4$ is tangent $\therefore$ P is (4,2)

$y = mx \Rightarrow 2 = 4m \Rightarrow m = \frac{1}{2} \Rightarrow 0.5$