JEE MAIN July 2021

MATHS

27th July 2021 [SHIFT – 2]

QUESTION WITH SOLUTION

JEE I NEET I Foundation

SECTION A

1. Which of the following is the negation of the statement "for all M>0, there exists x $\in$S such that x $\geq$ M"?

(1) there exists M > 0, such that x $\geq$ M for all x $\in$ S

(2) there exists M > 0, there exists x $\in$ S such that x $\geq$ M

(3) there exists M > 0, such that x < M for all x $\in$ S

(4) there exists M > 0, there exists x $\in$ S such that x < M

Sol. (3)

P : for all M > 0, there exists $x \in S$ such that $x \geq M$.

~ P : there exists M > 0, for all $x \in S$

Such that x < M

Negation of 'there exsits' is 'for all'.

2. For real numbers α and $\beta \neq 0$, if the point of intersection of the straight lines

$$\frac{x-\alpha}{1}=\frac{y-1}{2}=\frac{z-1}{3} \text{ and } \frac{x-4}{\beta}=\frac{y-6}{3}=\frac{z-7}{3},$$

lies on the plane x + 2y – z = 8, then $\alpha - \beta$ is equal to:

(1) 5 (2) 3 (3) 7 (4) 9

Sol. (3)

First line is $(\phi+\alpha, 2\phi+1, 3\phi+1)$

and second line is $(q\beta + 4, 3q + 6, 3q + 7)$.

For intersection $\phi + \alpha = q\beta + 4$...(i)

$2\phi + 1 = 3q + 6$. (i)

$3\phi + 1 = 3q + 7$. (iii)

for (ii) & (iii) $\phi = 1$, q = –1So, from (i) $\alpha + \beta = 3$

Now, point of intersection is $(\alpha + 1, 3, 4)$It lies on the plane.

Hence, $\alpha = 5$ & $\beta = -2$

3. The point P (a, b) undergoes the following three transformations successively:

(a) reflection about the line y = x.

(b) translation through 2 units along the positive direction of x-axis.

(c) rotation through angle $\frac{\pi}{4}$ about the origin in the anti-clockwise direction.

If the co-ordinates of the final position of the point P are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$, then the value of 2a + b is equal to:

(1) 9 (2) 5 (3) 13 (4) 7

Sol. (1)

Image of A(a,b) along y = x is B(b,a). Translating it 2 units it becomes C(b + 2, a).

Now, applying rotation theorem

$$-\frac{1}{\sqrt{2}}+\frac{7}{\sqrt{2}}i=((b+2)+ai)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$\frac{-1}{\sqrt{2}}+\frac{7}{\sqrt{2}}i=\left(\frac{b+2}{\sqrt{2}}-\frac{a}{\sqrt{2}}\right)+i\left(\frac{b+2}{\sqrt{2}}+\frac{a}{\sqrt{2}}\right)$$

$\Rightarrow$ b – a + 2 = -1 ......(1)

and b + 2 + a = 7 ....(2)

$\Rightarrow$ a = 4; b = 1

$\Rightarrow$ 2a + b = 9

4. Let $a=\max\limits_{x\in R}\{8^{2\sin 3x}.4^{4\cos 3x}\}$ and $\beta=\min\limits_{x\in R}\{8^{2\sin 3x}.4^{4\cos 3x}\}$

If $8x^2 + bx + c = 0$ is a quadratic equation whose roots are $\alpha^{1/5}$ and $\beta^{1/5}$, then the value of c – b is equal to:

(1) 43 (2) 42 (3) 50 (4) 47

Sol. (2)

$$\alpha = \max\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\}$$

$$=\max\{2^{6\sin 3x}\cdot 2^{8\cos 3x}\}$$

$$=\max\{2^{6\sin 3x+8\cos 3x}\}$$

and $\beta=\min\{8^{2\sin 3x}.4^{4\cos 3x}\}=\min\{2^{6\sin 3x+8\cos 3x}\}$

Now range of 6 sin 3x + 8 cos 3x

$$=\left[-\sqrt{6^2+8^2},+\sqrt{6^2+8^2}\right]=[-10,10]$$

$\alpha = 2^{10}$ & $\beta = 2^{-10}$

So, $\alpha^{1/5} = 2^2 = 4$

$\Rightarrow \beta^{1/5} = 2^{-2} = 1/4$

quadratic $8x^2 + bx + c = 0$, c – b =

$$8\times[(\text{product of roots})+(\text{sum of roots})]$$

$$=8\times\left[4\times\frac{1}{4}+4+\frac{1}{4}\right]=8\times\left[\frac{21}{4}\right]=42$$

5. Let f : R $\to$ R be defined as

$f(x + y) + f(x - y) = 2f(x) f(y)$, $f\left(\frac{1}{2}\right) = -1$. Then, the value of

$\sum\limits_{k=1}^{20}\frac{1}{\sin(k)\sin(k+f(k))}$ is equal to:

(1) $\text{cosec}^2(1)\ \text{cosec}(21)\sin(20)$ (2) $\sec^2(1)\sec(21)\cos(20)$

(3) $\text{cosec}^2(21)\cos(20)\cos(2)$ (4) $\sec^2(21)\sin(20)\sin(2)$

Sol. (1)

f(x) = cos ax

$\because\ f\left(\frac{1}{2}\right) = -1$

So, $-1 = \cos\frac{a}{2}$

$\Rightarrow\ a = 2(2n+1)\pi$

Thus f(x) = cos $2(2n+1)\pi x$

Now k is natural number

Thus f(k) = 1

$$\sum_{k=1}^{20}\frac{1}{\sin k \sin(k+1)} = \frac{1}{\sin 1}\sum_{k=1}^{20}\left[\frac{\sin((k+1)-k)}{\sin k\cdot\sin(k+1)}\right]$$

$$= \frac{1}{\sin 1}\sum_{k=1}^{20}(\cot k - \cot(k+1))$$

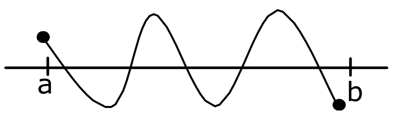
$$= \frac{\cot 1 - \cot 21}{\sin 1} = \operatorname{cosec}^2 1 \operatorname{cosec}(21)\cdot\sin 20$$

6. Let f: (a, b) → R be twice differentiable function such that f(x) = $\int_a^x g(t)dt$ for a differentiable function g(x). If f(x) = 0 has exactly five distinct roots in (a, b), then g(x) g′(x) = 0 has at least:

(1) seven roots in (a, b) (2) five roots in (a, b)

(3) three roots in (a, b) (4) twelve roots in (a, b)

Sol. (1)



$f(x) = \int_a^x g(t)\,dt$

$f(x) \to 5$

$f'(x) \to 4$

$g(x) \to 4$

$g'(x) \to 3$

7. Let C be the set of all complex numbers. Let

$S_1 = \{z \in C : |z-2| \le 1\}$ and

$S_2 = \{z \in C : z(1+i) + \bar{z}(1-i) \ge 4\}$

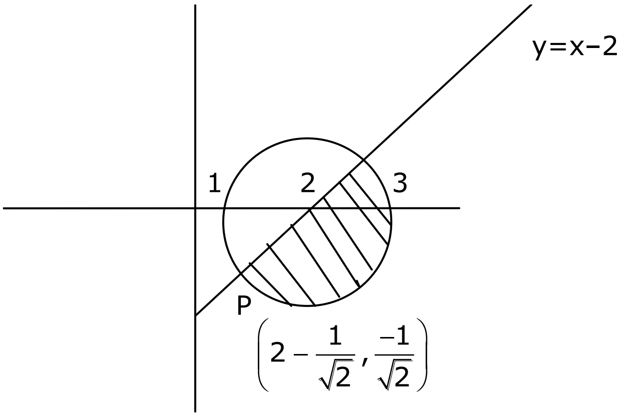
Then, the maximum value of $\left|z-\frac{5}{2}\right|^2$ for $z \in S_1 \cap S_2$ is equal to:

(1) $\frac{3+2\sqrt{2}}{4}$ (2) $\frac{5+2\sqrt{2}}{2}$ (3) $\frac{3+2\sqrt{2}}{2}$ (4) $\frac{5+2\sqrt{2}}{4}$

Sol. **(4)**

$S_1 : |z-2| \le 1$

$S_2 : x(1+i) + \bar{z}(1-i) \ge 4$



$z = x + iy$

$(x+iy)(1+i) + (x-iy)(1-i) \ge 4$

$x + i(x+y) - y + x - i(x+y) - y \ge 4$

$2x - 2y \ge 4$

$\Rightarrow \;\; y \le x - 2$

For P $(x-2)^2 + (x-2)^2 = 1$

$\Rightarrow \;\; x = 2 - \frac{1}{\sqrt{2}}, 2 + \frac{1}{\sqrt{2}}$

$\therefore$ from diagram x-coordinate of $P = 2 - \frac{1}{\sqrt{2}}$

$\therefore \;\; P = (2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$\therefore \;\; \left|Z - \frac{5}{2}\right|^2_{max} = \left|2 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i - \frac{5}{2}\right|^2$

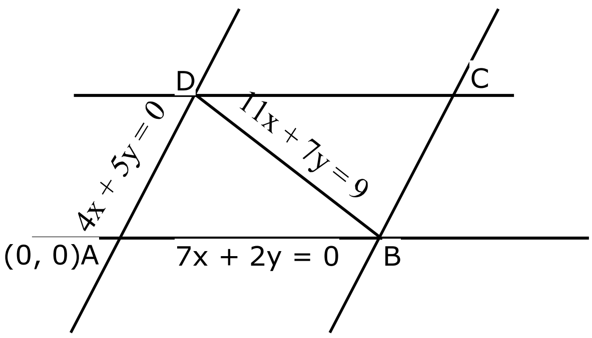
$= \left|\left(\frac{-1}{2} - \frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}\right|^2 = \frac{5+2\sqrt{2}}{4}$

8. Two sides of a parallelogram are along the lines 4x + 5y = 0 and 7x + 2y = 0. If the equation of one of the diagonals of the parallelogram is 11x + 7y = 9, then other diagonal passes through the point:

(1) (1, 3) (2) (1, 2) (3) (2, 2) (4) (2, 1)

Sol. **(3)**

Both the lines pass through origin.



point D is equal of intersection of $4x + 5y = 0$ & $11x + 7y = 9$

So, coordinates of point D = $\left(\frac{5}{3}, -\frac{4}{3}\right)$

Also, point B is point of intersection of $7x + 2y = 0$ & $11x + 7y = 9$

So, coordinates of point B = $\left(-\frac{2}{3}, \frac{7}{3}\right)$

diagonals of parallelogram intersect at middle let middle point of B,D

$$\Rightarrow \left(\frac{\frac{5}{3}-\frac{2}{3}}{2}, \frac{\frac{-4}{3}+\frac{7}{3}}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

equation of diagonal AC

$$\Rightarrow (y-0) = \frac{\frac{1}{2}-0}{\frac{1}{2}-0}(x-0)$$

$y = x$

diagonal AC passes through (2, 2).

9. Let $f : [0,\infty) \to [0, 3]$ be a function defined by

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x\}, & 0 \le x \le \pi \\ 2 + \cos x, & x > \pi \end{cases}$$

Then which of the following is true?

(1) f is differentiable everywhere in $(0, \infty)$

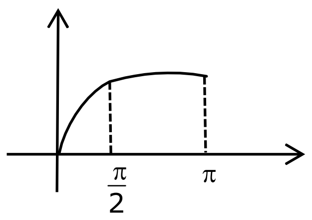
(2) f is continuous everywhere but not differentiable exactly at two points in $(0, \infty)$

(3) f is not continuous exactly at two points in $(0, \infty)$

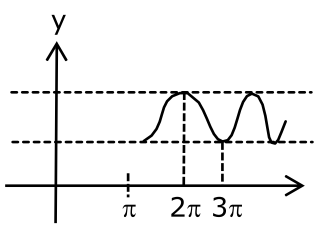
(4) f is continuous everywhere but not differentiable exactly at one point in $(0, \infty)$

Sol. (1)

Graph of max{ $\sin t$: $0 \le t \le x$ } in $x \in [0, \pi]$

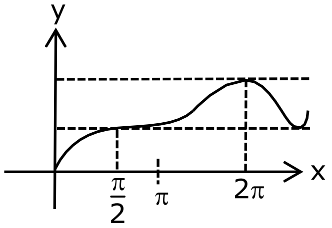


& graph of cos for $x \in [\pi, \infty)$



So graph of

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x, & 0 \le x \le \pi \\ 2 + \cos x & x > \pi \end{cases}$$

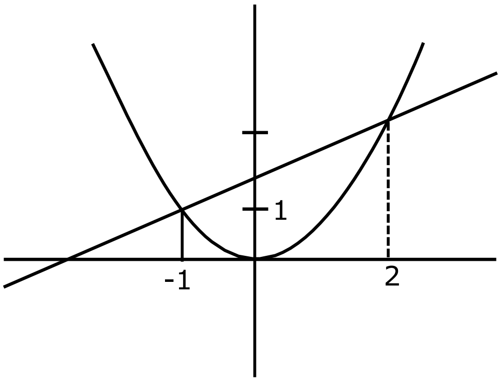


f(x) is differentiable everywhere in $(0, \infty)$

10. The area of the region bounded by y – x = 2 and x^2 = y is equal to :

(1) $\frac{4}{3}$ (2) $\frac{9}{2}$ (3) $\frac{16}{3}$ (4) $\frac{2}{3}$

Sol. (2)



y – x = 2 , x^2 = y

Now, x^2 = 2 + x

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x + 1)(x - 2) = 0$

$$\text{Area} = \int_{-1}^{2} (2 + x - x^2)$$

$$= \left| 2x + \frac{x^2}{2} - \frac{x^3}{3} \right|_{-1}^{2}$$

$$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 6 - 3 + 2 - \frac{1}{2} = \frac{9}{2}$$

11. Let the mean and variance of the frequency distribution

x:	$x_1 = 2$	$x_2 = 6$	$x_3 = 8$	$x_4 = 9$
f:	4	4	α	β

be 6 and 6.8 respectively. If x_3 is changed from 8 to 7, then the mean for the new data will be:

(1) $\frac{16}{3}$ (2) 4 (3) $\frac{17}{3}$ (4) 5

Sol. **(3)**

Given $32 + 8\alpha + 9\beta = (8 + \alpha + \beta) \times 6$

$\Rightarrow 2\alpha + 3\beta = 16$ ...(i)

Also, $4 \times 16 + 4 \times \alpha + 9\beta = (8 + \alpha + \beta) \times 6.8$

$\Rightarrow 640 + 40\alpha + 90\beta = 544 + 68\alpha + 68\beta$

$\Rightarrow 28\alpha - 22\beta = 96$

$\Rightarrow 14\alpha - 11\beta = 48$ ...(ii) from (i) & (ii)

$\alpha = 5$ & $\beta = 2$

so, new mean = $\dfrac{32+35+18}{15} = \dfrac{85}{15} = \dfrac{17}{3}$

12. A possible value of 'x', for which the ninth term in the expansion of

$\left\{3^{\log_3\sqrt{25^{x-1}+7}} + 3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}\right\}^{10}$ in the increasing powers of $3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}$

is equal to 180, is:

(1) 2 (2) 1 (3) 0 (4) –1

Sol. **(2)**

$^{10}C_8(25^{(x-1)} + 7) \times (5^{(x-1)} + 1)^{-1} = 180$

$\Rightarrow \dfrac{25^{x-1}+7}{5^{(x-1)}+1} = 4$

$\Rightarrow \dfrac{t^2+7}{t+1} = 4;$

$\Rightarrow t = 1, 3 = 5^{x-1}$

$\Rightarrow x - 1 = 0$ (one of the possible value).

$\Rightarrow x = 1$

13. Let $y = y(x)$ be the solution of the differential equation $(x - x^3)dy = (y + yx^2 - 3x^4)dx$, $x > 2$. If $y(3) = 3$, then $y(4)$ is equal to :

(1) 8 (2) 12 (3) 16 (4) 4

Sol. **(2)**

$(x - x^3)dy = (y + yx^2 - 3x^4)dx$

$\Rightarrow xdy - ydx = (yx^2 - 3x^4)dx + x^3dy$

$\Rightarrow \dfrac{xdy - ydx}{x^2} = (ydx + xdy) - 3x^2\,dx$

$\Rightarrow d\left(\dfrac{y}{x}\right) = d(xy) - d(x^3)$

Integrate

$\dfrac{y}{x} = xy - x^3 + c$

$y(3) = 3$

$\Rightarrow \frac{3}{3} = 3\times3-3^3+c$

$\Rightarrow c = 19$

$\therefore \frac{y}{x} = xy - x^3 + 19$

at x = 4, $\frac{y}{4} = 4y - 64 + 19$

$15y = 4\times45$

$\Rightarrow$ y = 12

14. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} = \vec{b} \times (\vec{b}\times\vec{c})$. If magnitudes of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are $\sqrt{2}$, 1 and 2 respectively and the angle between $\vec{b}$ and $\vec{c}$ is $\theta\left(0<\theta<\frac{\pi}{2}\right)$, then the value of 1+ $\tan\theta$ is equal to:

(1) $\frac{\sqrt{3}+1}{\sqrt{3}}$ (2) 2 (3) $\sqrt{3}+1$ (4) 1

Sol. (2)

$\vec{a} = (\vec{b}\cdot\vec{c})\vec{b} - (\vec{b}.\vec{b})\vec{c}$

$= 1.2\cos\theta\vec{b} - \vec{c}\,b$

$\Rightarrow \vec{a} = 2\cos\theta\vec{b} - \vec{c}$

$|\vec{a}|^2 = (2\cos\theta)^2 + 2^2 - 2.2\cos\theta\vec{b}\cdot\vec{c}$

$\Rightarrow 2 = 4\cos^2\theta + 4 - 4\cos\theta\cdot2\cos\theta$

$\Rightarrow -2 = -4\cos^2\theta$

$\Rightarrow \cos^2\theta = \frac{1}{2}$

$\Rightarrow \sec^2\theta = 2$

$\Rightarrow \tan^2\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

$1+\tan\theta = 2$.

15. A student appeared in an examination consisting of 8 true – false type questions. The student guesses the answers with equal probability. The smallest value of n, so that the probability of guessing at least 'n' correct answers is less than $\frac{1}{2}$, is:

(1) 5 (2) 3 (3) 6 (4) 4

Sol. (1)

$P(E) < \frac{1}{2}$

$$\Rightarrow \sum_{r=n}^{8} {}^8C_r \left(\frac{1}{2}\right)^{8-r}\left(\frac{1}{2}\right)^{r} < \frac{1}{2}$$

$$\Rightarrow \sum_{r=n}^{8} {}^8C_r \left(\frac{1}{2}\right)^{8} < \frac{1}{2}$$

$$\Rightarrow {}^8C_n + {}^8C_{n+1} + ... + {}^8C_8 < 128$$

$$\Rightarrow 256 - ({}^8C_0 + {}^8C_1 + ... + {}^8C_{n-1}) < 128$$

$$\Rightarrow {}^8C_0 + {}^8C_1 + ... + {}^8C_{n-1} < 128$$

$$\Rightarrow n - 1 \geq 4$$

$$\Rightarrow n \geq 5$$

16. If $\tan\left(\frac{\pi}{9}\right)$, x, $\tan\left(\frac{7\pi}{18}\right)$ are in arithmetic progression and $\tan\left(\frac{\pi}{9}\right)$, y, $\tan\left(\frac{5\pi}{18}\right)$ are also in arithmetic progression, then $|x - 2y|$ is equal to:

(1) 0 (2) 3 (3) 4 (4) 1

Sol. **(1)**

$$x = \frac{1}{2}\left(\tan\frac{\pi}{9} + \tan\frac{7\pi}{18}\right)$$

and $2y = \tan\frac{\pi}{9} + \tan\frac{5\pi}{18}$

so, $x - 2y = \frac{1}{2}\left(\tan\frac{\pi}{9} + \tan\frac{7\pi}{18}\right)$

$$-\left(\tan\frac{\pi}{9} + \tan\frac{5\pi}{18}\right)$$

$$\Rightarrow |x - 2y| = \left|\frac{\cot\frac{\pi}{9} - \tan\frac{\pi}{9}}{2} - \tan\frac{5\pi}{18}\right|$$

$$= \left|\cot\frac{2\pi}{9} - \cot\frac{2\pi}{9}\right| = 0$$

$$\left(\text{as} \tan\frac{5\pi}{18} = \cot\frac{2\pi}{9}; \tan\frac{7\pi}{18} = \cot\frac{\pi}{9}\right)$$

17. Let N be the set of natural numbers and a relation R on N be defined by $R = \{(x, y) \in N \times N : x^3 - 3x^2y - xy^2 + 3y^3 = 0\}$. Then the relation R is:

(1) reflexive and symmetric, but not transitive

(2) reflexive but neither symmetric nor transitive

(3) an equivalence relation

(4) symmetric but neither reflexive nor transitive

Sol. **(2)**

$x^3 - 3x^2y - xy^2 + 3y^3 = 0$

$\Rightarrow x(x - y)(x + y) - 3y(x - y)(x + y) = 0$

$\Rightarrow (x - 3y)(x - y)(x + y) = 0$

Now, $x = y\ \forall (x,y) \in N \times N$ so reflexive

But not symmetric & transitive

See, (3,1) satisfies but (1,3) does not.

Also (3,1) &(1,–1) satisfies but (3, –1) does not

18. The value of $\lim_{x\to 0}\left(\dfrac{x}{\sqrt[8]{1-\sin x}-\sqrt[8]{1+\sin x}}\right)$ is equal to:

(1) –1 (2) –4 (3) 0 (4) 4

Sol. **(2)**

Rationalize denominator three times

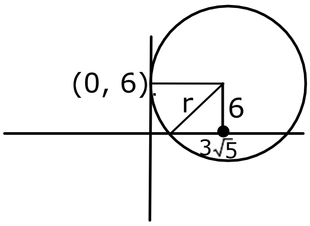
$$\lim_{x\to 0}\frac{x\left\{(1-\sin x)^{1/8}+(1+\sin x)^{1/8}\right\}\left\{(1-\sin x)^{1/4}+(1+\sin x)^{1/4}\right\}\left\{(1-\sin x)^{1/2}+(1+\sin x)^{1/2}\right\}}{(1-\sin x-1-\sin x)}$$

$$\lim_{x\to 0}\frac{8x}{-2\sin x}=-4$$

19. Consider a circle C which touches the y-axis at (0, 6) and cuts off an intercept $6\sqrt{5}$ on the x-axis. Then the radius of the circle C is equal to:

(1) $\sqrt{82}$ (2) 9 (3) 8 (4) $\sqrt{53}$

Sol. **(2)**



$r = \sqrt{6^2 + (3\sqrt{5})^2}$

$= \sqrt{36+45} = 9$

20. Let A and B be two 3 × 3 real matrices such that $(A^2 - B^2)$ is invertible matrix. If $A^5 = B^5$ and $A^3B^2 = A^2B^3$, then the value of the determinant of the matrix $A^3 + B^3$ is equal to:

(1) 0 (2) 2 (3) 1 (4) 4

Sol. **(1)**

$C = A^2 - B^2; |C| \neq 0$

$A^5 = B^5$ and $A^3B^2 = A^2B^3$

Now, $A^5 - A^3B^2 = B^5 - A^2B^3$

$\Rightarrow A^3(A^2 - B^2) + B^3(A^2 - B^2) = 0$

$\Rightarrow (A^3 + B^3)(A^2 - B^2) = 0$

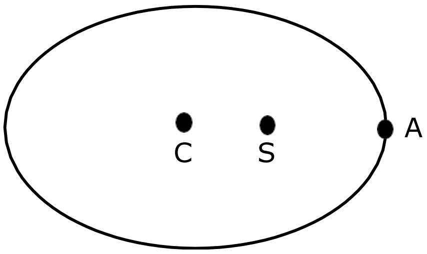
Post multiplying inverse of $A^2 - B^2$:

$A^3 + B^3 = 0$

SECTION B

1. Let E be an ellipse whose axes are parallel to the co-ordinates axes, having its center at (3, –4), one focus at (4, –4) and one vertex at (5, –4). If $mx - y = 4$, $m > 0$ is a tangent to the ellipse E, then the value of $5m^2$ is equal to _____.

Sol. **(3)**



and A(5,–4)

Hence, a = 2 & ae = 1

$\Rightarrow e = \dfrac{1}{2}$

$\Rightarrow b^2 = 3.$

So, $E: \dfrac{(x-3)^2}{4} + \dfrac{(y+4)^2}{3} = 1$

Intersecting with given tangent.

$\dfrac{x^2 - 6x + 9}{4} + \dfrac{m^2x^2}{3} = 1$

Now, $D = 0$ (as it is tangent)

So, $5m^2 = 3$.

2. If the real part of the complex number $z = \dfrac{3 + 2i\cos\theta}{1 - 3i\cos\theta}$, $\theta \in \left(0, \dfrac{\pi}{2}\right)$ is zero, then the value of $\sin^2 3\theta + \cos^2\theta$ is equal to _______.

Sol. **(1)**

$\text{Re}\ (z) = \dfrac{3 - 6\cos^2\theta}{1 + 9\cos^2\theta} = 0$

$\Rightarrow \theta = \dfrac{\pi}{4}$

Hence, $\sin^2 3\theta + \cos^2\theta = 1$.

3. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $M = A + A^2 + A^3 + ... + A^{20}$, then the sum of all the elements of the matrix M is equal to ________.

Sol. (2020)

$$A^n = \begin{bmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

So, required sum

$$= 20 \times 3 + 2 \times \left(\frac{20 \times 21}{2}\right) + \sum_{r=1}^{20}\left(\frac{r^2+r}{2}\right)$$

= 60 + 420 + 105 + 35 × 41 = 2020

4. The distance of the point P(3, 4, 4) from the point of intersection of the line joining the points Q(3, −4, −5) and R(2, −3, 1) and the plane 2x +y + z = 7, is equal to ______.

Sol. (7)

$$\overline{QR} : \frac{x-3}{1} = \frac{y+4}{-1} = \frac{z+5}{-6} = r$$

$$\Rightarrow (x, y, z) \equiv (r+3, -r-4, -6r-5)$$

Now, satisfying it in the given plane.

$$2(r+3) + (-r-4) + (-6r-5) = 7$$

We get r = -2.

so, required point of intersection is T(1, -2,7).

Hence, PT = 7.

5. The number of real roots of the equation $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$ is equal to ______.

Sol. (2)

$t^4 - t^3 - 4t^2 - t + 1 = 0$, $e^x = t > 0$

$$\Rightarrow t^2 - t - 4 - \frac{1}{t} + \frac{1}{t^2} = 0$$

$$\Rightarrow \alpha^2 - \alpha - 6 = 0, \alpha = t + \frac{1}{t} \geq 2$$

$$\Rightarrow \alpha = 3, -2(\text{reject})$$

$$\Rightarrow t + \frac{1}{t} = 3$$

$\Rightarrow$ The number of real roots = 2

6. Let n be a non-negative integer. Then the number of divisors of the form "4n + 1" of the number $(10)^{10}.\ (11)^{11}.\ (13)^{13}$ is equal to ________.

Sol. (924)

$N = 2^{10} \times 5^{10} \times 11^{11} \times 13^{13}$

Now, power of 2 must be zero,

power of 5 can be anything,

power of 13 can be anything. But,

power of 11 should be even.

So, required number of divisors is

1 × 11 × 14 × 6 = 924

7. If $\int_0^{\pi}\left(\sin^3 x\right)e^{-\sin^2 x}dx = \alpha - \frac{\beta}{e}\int_0^1 \sqrt{t}\ e^t\,dt$, then $\alpha + \beta$ is equal to _____.

Sol. **(5)**

$$I = 2\int_0^{\pi/2} \sin^3 x e^{-\sin^2 x}dx$$

$$= 2\int_0^{\pi/2} \sin x\, e^{-\sin^2 x}dx + \int_0^{\pi/2} \underset{\mathrm{I}}{\cos x}\,\underbrace{e^{-\sin^2 x}(-\sin 2x)}_{\mathrm{II}}dx$$

$$= 2\int_0^{\pi/2} \sin x\, e^{-\sin^2 x}dx + \left[\cos x e^{-\sin^2 x}\right]_0^{\pi/2}$$

$$+\int_0^{\pi/2} \sin x\, e^{-\sin^2 x}dx$$

$$= 3\int_0^{\pi/2} \sin x e^{-\sin^2 x}dx \quad -1$$

$$= \frac{3}{2}\int_{-1}^{0} \frac{e^{\alpha}d\alpha}{\sqrt{1+\alpha}} - 1(\text{Put} - \sin^2 x = t)$$

$$= \frac{3}{2}\int_{-1}^{0} \frac{e^{\alpha}d\alpha}{\sqrt{1+\alpha}} - 1(\text{put}\, 1+\alpha = x)$$

$$= \frac{3}{2e}\int_{-1}^{0} \underset{\mathrm{I}}{e^x}\,\underset{\mathrm{II}}{\frac{1}{\sqrt{x}}}dx - 1\ b$$

$$= 2 - \frac{3}{e}\int_0^1 e^x\sqrt{x}dx$$

Hence, $\alpha + \beta = 5$

8. Let $\vec{a} = \hat{i} - \alpha\hat{j} + \beta\hat{k}$, $\vec{b} = 3\hat{i} + \beta\hat{j} - \alpha\hat{k}$ and $\vec{c} = -\alpha\hat{i} - 2\hat{j} + \hat{k}$, where α and β are integers. If $\vec{a}.\vec{b} = -1$ and $\vec{b}.\vec{c} = 10$, then $\left(\vec{a}\times\vec{b}\right).\vec{c}$ is equal to ______.

Sol. **(9)**

$\vec{a} = (1, -\alpha, \beta)$

$\vec{b} = (3, \beta, -\alpha)$

$\vec{c} = (-\alpha, -2, 1);\ \alpha, \beta \in I$

$\vec{a}.\vec{b} = -1 \Rightarrow 3 - \alpha\beta - \alpha\beta = -1$

$\Rightarrow \alpha\beta = 2$

$\vec{b}.\vec{c} = 10$

$\Rightarrow -3\alpha - 2\beta - \alpha = 10$

$\Rightarrow 2\alpha + \beta + 5 = 0$

$\therefore \alpha = -2; \beta = -1$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 1 \end{vmatrix}$$

$= 1(-1 + 4) - 2\,(3 - 4) - 1(-6 + 2)$

$= 3 + 2 + 4 = 9$

9. Let A = $\{n \in N \mid n^2 \le n + 10{,}000\}$, B = $\{3k+1 \mid k \in N\}$ and C = $\{2k \mid k \in N\}$, then the sum of all the elements of the set $A \cap (B - C)$ is equal to ________.

Sol. **(832)**

$B - C \equiv \{7, 13, 19, ...97, ...\}$

Now, $n^2 - n \le 100 \times 100$

$\Rightarrow n(n - 1) \le 100 \times 100$

$\Rightarrow A = \{1, 2, ..., 100\}$.

So, $A \cap (B - C) = \{7, 13, 19, ..., 97\}$

Hence, $\text{sum} = \frac{16}{2}(7 + 97) = 832$.

10. Let y = y(x) be the solution of the differential equation $dy = e^{\alpha x + y}dx$; $\alpha \in N$. If $y(\log_e 2) = \log_e 2$ and $y(0) = \log_e\left(\frac{1}{2}\right)$, then the value of α is equal to ________.

Sol. **(2)**

$$\int e^{-y}dy = \int e^{\alpha x}dx$$

$$\Rightarrow -e^{-y} = \frac{e^{\alpha x}}{\alpha} + c \qquad ....(i)$$

Put $(x, y) = (\ell n2, \ell n2)$

$$\frac{-1}{2} = \frac{2^{\alpha}}{\alpha} + C \qquad ....(ii)$$

Put $(x, y) \equiv (0, -\ell n2)$ in (i)

$$-2 = \frac{1}{\alpha} + C \qquad ....(iii)$$

(ii) – (iii)

$$\frac{2\alpha - 1}{\alpha} = \frac{3}{2}$$

$\Rightarrow \alpha = 2 (\text{as } \alpha \in N)$