JEE MAIN July 2021

MATHS

25th July 2021 [SHIFT – 1]

QUESTION WITH SOLUTION

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SECTION - A

1. Let $f:[0,\infty)\to[0,\infty)$ be defined as

$f(x)\ \int_0^x [y]dy$

Where [x] is the greatest integer less than or equal to x. Which of the following is true?

(1) f is differentiable at every point in $[0,\infty)$.

(2) f is continuous everywhere except at the integer points in $[0,\infty)$.

(3) f is continuous at every point in $[0,\infty)$ and differentiable except at the integer points.

(4) f is both continuous and differentiable except at the integer points in $[0,\infty)$.

Sol. (3)

$f:[0,\infty)\to[0,\infty), f(x)=\int_0^x [y]dy$

Let x = n + f, f $\in$ (0, 1)

So f(x) = 0 + 1 + 2 + ... + (n – 1) + $\int_n^{n+f} n\,dy$

$f(x) = \frac{n(n-1)}{2}+nf$

$= \frac{[x]([x]-1)}{2}+[x]\{x\}$

Note $\lim_{x\to n^+} f(x)=\frac{n(n-1)}{2}$, $\lim_{x\to n^-} f(x)=\frac{(n-1)(n-2)}{2}+(n-1)$

$=\frac{n(n-1)}{2}$

$f(x) = \frac{n(n-1)}{2}(n\in N_0)$

so f(x) is cont. $\forall$ x ≥ 0 and diff. except at integer points

2. Let g : N $\to$ N be defined as

g(3n+1) = 3n + 2,

g(3n+2) = 3n + 3,

g(3n+3) = 3n+1, for all n ≥ 0.

Then which of the following statements is true?

(1) gogog = g

(2) There exists an onto function f: N $\to$ N such that fog = f

(3) There exists a one- one function f: N $\to$ N such that fog = f

(4) There exists an function f: N $\to$ N such that gof = f

Sol. (2)

g(3n +1) = 3n + 2

g(3n + 2) = 3n + 3

g(3n + 3) = 3n + 1, n $\geq$ 0

For x = 3n + 1

(1) gogog (3n+1) = gog(3n+2) = g(3n+3)=3n+1

Similarly

gogog (3n+2) = 3n+2

gogog (3n+3) = 3n+3

So gogog (x) = x $\forall$ $x \in N$

(2) If $f:N \to N$ and f is an onto function such that f(g(x)) = f(x) then

One of its possibilities is by taking f(x) as onto function

$$f(x) = \begin{cases} a & x = 3n+1 \\ a & x = 3n+2, \quad a \in N \\ a & x = 3n+3 \end{cases}$$

(3) If f : N $\to$ N and f is a one-one function such that f(g(x)) = f(x) then

g(x) = x

but $g(x) \neq x$

(4) As f : N $\to$ N, f=3n+1

=3n+2

So, g(3n+1) = 3n+2, g(3n+2)=3n+3, g(3n+3) = 3n+1

So g(f(x)) $\neq$ f(x)

3. Let 9 distinct balls be distributed among 4 boxes, B_1, B_2, B_3 and B_4. If the probability that B_3 contains exactly 3 balls is $k\left(\frac{3}{4}\right)^9$ then k lies in the set:

(1) $\{x \in R : |x-5| \le 1\}$ (2) $\{x \in R : |x-2| \le 1\}$ (3) $\{x \in R : |x-3| < 1\}$ (4) $\{x \in R : |x-1| < 1\}$

Sol. (3)

required probability = $\frac{{}^9C_3 . 3^6}{4^9}$

$= \frac{{}^9C_3}{27} . \left(\frac{3}{4}\right)^9$

$= \frac{28}{9} . \left(\frac{3}{4}\right)^9 \Rightarrow k = \frac{28}{9}$

Which satisfies $|x-3| < 1$

4. The Boolean expression (p $\Rightarrow$ q)^ (q$\Rightarrow$ ~p) is equivalent to :

(1) q (2) ~q (3) p (4) ~p

Sol. (4)

(p $\to$ q) $\wedge$ (q $\to$~p)

=(~Pvq) $\wedge$ (~qv~p) {P$\to$q$\equiv$~p v q}

$\equiv$(~p v q) $\wedge$ (~pv~q) {commutative property}

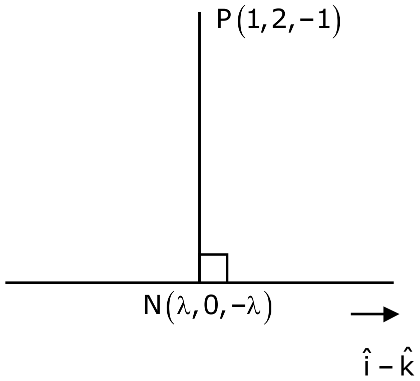
$\equiv$~p v (qv~q) {distributive property}

$\equiv$ ~p

5. Let the foot of perpendicular from a point P(1,2,-1) to the straight line L: $\frac{x}{1}=\frac{y}{0}=\frac{z}{-1}$ be N. Let a line be drawn from P parallel to the plane x + y + 2z = 0 which meets L at point Q. If α is the acute angle between the lines PN and PQ, then cos α is equal to______

(1) $\frac{1}{2\sqrt{3}}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{\sqrt{3}}{2}$ (4) $\frac{1}{\sqrt{3}}$

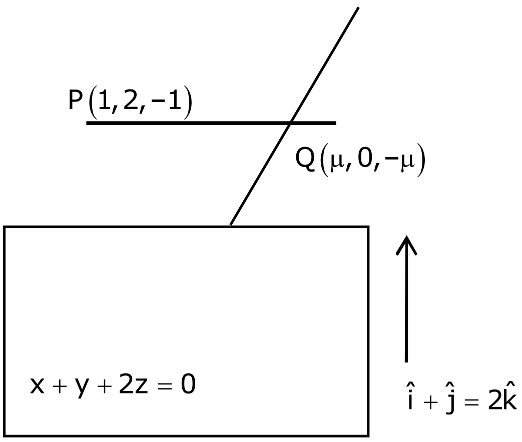
Sol. (4)



$\overrightarrow{PN}.\left(\hat{i}-\hat{k}\right) = 0$

$\Rightarrow$ N(1, 0, –1)

Now,



$\overrightarrow{PQ}.\left(\hat{i}+\hat{j}+2\hat{k}\right) = 0$

$\Rightarrow$ m = –1

$\Rightarrow$ Q (–1, 0, 1)

$\overrightarrow{PN}=2\hat{j}$ and $\overrightarrow{PQ}=2\hat{i}+2\hat{j}-2\hat{k}$

$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$

6. Let f(x) = $3\sin^4x + 10\sin^3x + 6\sin^2x - 3$, $x \in \left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$. Then, f is

(1) increasing in $\left(-\frac{\pi}{6}, 0\right)$ (2) decreasing in $\left(0, \frac{\pi}{2}\right)$

(3) decreasing in $\left(-\frac{\pi}{6}, 0\right)$ (4) increasing in $\left(-\frac{\pi}{6}, \frac{\pi}{2}\right)$

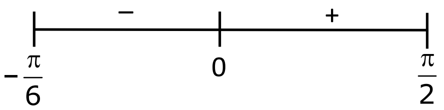
Sol. (3)

$f(x) = 3\sin^4x + 10\sin^3x + 6\sin^2x - 3,\ x \in \left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$

$f(x) = 12\sin^3x\cos x + 30\sin^2x\cos x + 12\sin x\cos x$

$= 6\sin x\cos x\,(2\sin^2x + 5\sin x + 2)$

$= 6\sin x\cos x\,(2\sin x + 1)(\sin + 2)$



Dressing in $\left(-\frac{\pi}{6}, 0\right)$

7. Let S_n be the sum of the first n terms of an arithmetic progression. If $S_{3n} = 3S_{2n}$, then the value of $\frac{S_{4n}}{S_{2n}}$ is:

(1) 4 (2) 6 (3) 8 (4) 2

Sol. (2)

Let a be first term and d be common diff. of this A.P.

Given $S_{3n} = 3S_{2n}$

$\Rightarrow \frac{3n}{2}\left[2a + (3n-1)d\right] = 3\frac{2n}{2}\left[2a + (2n-1)d\right]$

$\Rightarrow 2a + (3n-1)d = 4a + (4n-2)d$

$\Rightarrow 2a + (n-1)d = 0$

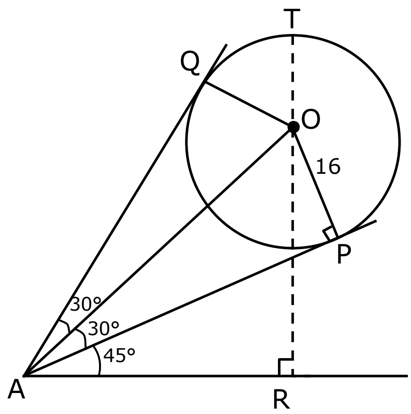
Now $\frac{S_{4n}}{S_{2n}} = \frac{\frac{4n}{2}\left[2a + (4n-1)d\right]}{\frac{2n}{2}\left[2a + (2n-1)d\right]} = \frac{2\left[\underbrace{2a + (n-1)d}_{=0} + 3nd\right]}{\left[\underbrace{2a + (n-1)d}_{=0} + nd\right]}$

$= \frac{6nd}{nd} = 6$

8. A spherical gas balloon of radius 16 meter subtends an angle 60° at the eye of the observer A while the angle of elevation of its center from the eye of A is 75°. Then the height (in meter) of the top most point of the balloon from the level of the observer's eye is:

(1) $8\left(\sqrt{2} + 2 + \sqrt{3}\right)$ (2) $8\left(\sqrt{6} + \sqrt{2} + 2\right)$ (3) $8\left(2 + 2\sqrt{3} + \sqrt{2}\right)$ (4) $8\left(\sqrt{6} - \sqrt{2} + 2\right)$

Sol. (2)



O $\rightarrow$ centre of sphere

P, Q $\rightarrow$ point of contact of tangent from A

Let T be top most point of balloon & R be foot of perpendicular from O to ground.

From triangle OAP, OA = 16cosec30° = 32

From triangle ABO, Or = OA sin75° = 32 $\dfrac{\left(\sqrt{3}+1\right)}{2\sqrt{2}}$

So level of top most point = OR + OT

$= 8\left(\sqrt{6}+\sqrt{2}+2\right)$

9. If b is very small as compared to the value of a, so that the cube and other higher powers of $\dfrac{b}{a}$ can be neglected in the identity $\dfrac{1}{a-b}+\dfrac{1}{a-2b}+\dfrac{1}{a-3b}\ldots\ldots+\dfrac{1}{a-nb}=\alpha n+\beta n^2+\gamma n^3$, then the value of γ is:

(1) $\dfrac{b^2}{3a^3}$ (2) $\dfrac{a+b}{3a^2}$ (3) $\dfrac{a^2+b}{3a^3}$ (4) $\dfrac{a+b^2}{3a^3}$

Sol. (1)

$(a-b)^{-1}+(a-2b)^{-1}+\ldots\ldots+(a-nb)^{1}$

$= \dfrac{1}{a}\sum_{r=1}^{n}\left(1-\dfrac{rb}{a}\right)^{-1}$

$= \dfrac{1}{a}\sum_{r=1}^{n}\left\{\left(1+\dfrac{rb}{a}+\dfrac{r^2b^2}{a^2}\right)+(\text{terms to be neglected})\right\}$

$= \dfrac{1}{a}\left[n+\dfrac{n(n+1)}{2}\cdot\dfrac{b}{a}+\dfrac{n(n+1)(2n+1)}{6}\cdot\dfrac{b^2}{a^2}\right]$

$= \dfrac{1}{a}\left[n^3\left(\dfrac{b^2}{3a^2}\right)+\ldots\right]$

So $\gamma=\dfrac{b^2}{3a^2}$

10. Let the vectors

$(2+a+b)\hat{i} + (a+2b+c)\hat{j} - (b+c)\hat{k}$

$(1+b)\hat{i} + 2b\hat{j} - b\hat{k}$ and $(2+b)\hat{i} + 2b\hat{j} + (1-b)\hat{k}$ a, b, c $\in$R be co-planar. Then which of the following is true?

(1) 2a = b + c (2) 2b = a + c (3)3c = a + b (4) a = b + 2c

Sol. (2)

If the vectors are co-planner,

$$\begin{vmatrix} a+b+2 & a+2b+C & -b-c \\ b+1 & 2b & -b \\ b+2 & 2b & 1-b \end{vmatrix} = 0$$

Now $R_3 \to R_3 - R_2$, $R_1 \to R_1 - R_2$

So $\begin{vmatrix} a+1 & a+c & -c \\ b+1 & 2b & -b \\ 1 & 0 & 1 \end{vmatrix} = 0$

$= (a + 1)\, 2b - (a + c)(2b + 1) - c(-2b)$

$= 2ab + 2b - 2ab - a - 2bc - c + 2bc$

$= 2b - a - c = 0$

11. Let f: R $\to$ R be defined as

$$f(x)\begin{cases} \dfrac{\lambda\,|x^2-5x+6|}{\mu(5x-x^2-6)}, & x<2 \\ e^{\frac{\tan(x-2)}{x-[x]}}, & x>2 \\ \mu & x=2 \end{cases}$$

Where [x] is the greatest integer less than or equal to x. If f is continuous at x =2, then $\lambda+\mu$ is equal to:

(1) e(e-2) (2) 2e-1 (3) e(-e+1) (4) 1

Sol. (3)

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} e^{\frac{\tan(x-2)}{x-2}} = e^1$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{-\lambda(x-2)(x-3)}{\mu(x-2)(x-3)} = -\frac{\lambda}{\mu}$$

For continuity $\mu = e = -\dfrac{\lambda}{\mu} \Rightarrow \mu = e,\ \lambda = -e^2$

$\lambda + \mu = e(-e + 1)$

12. The number of real roots of the equation $e^{6x}-e^{4x}-2e^{3x} - 12e^{2x}+e^{x}+1 =0$ is:

(1) 1 (2) 6 (3) 4 (4) 2

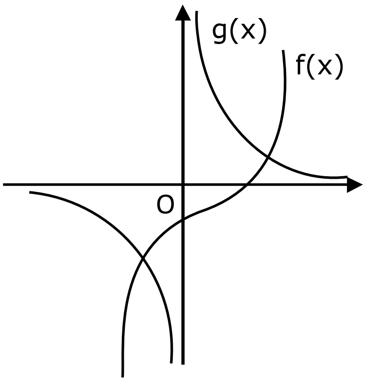
Sol. (4)

$e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^{x} + 1 = 0$

$\Rightarrow (e^{3x} - 1)^2 - e^{x}(e^{3x} - 1) = 12e^{2x}$

$(e^{3x} - 1)^2 (e^{x} - e^{-x} - e^{-2x}) = 12$

$\Rightarrow \underbrace{e^{x} - e^{-x} - e^{-2x}}_{\text{increasing(let } f(x))} = \frac{12}{\underbrace{e^{3x} - 1}_{\text{decreasing (let } g(x))}}$

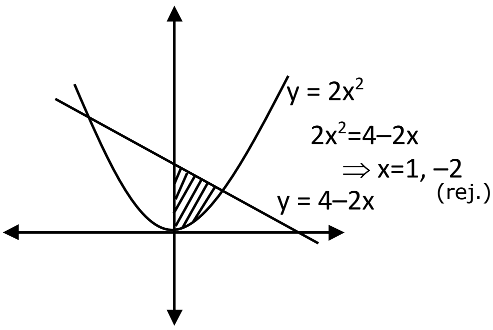


$\Rightarrow$ No. of real roots = 2

13. The area (in sq. units) of the region, given by the set $\{(x,y) \in R \times R \mid x \geq 0, 2x^2 \leq y \leq 4 - 2x\}$ is:

(1) $\frac{7}{3}$ (2) $\frac{13}{3}$ (3) $\frac{17}{3}$ (4) $\frac{8}{3}$

Sol. (1)



Required area $= \int_0^1 (4 - 2x - 2s^2)\,dx = 4x - x^2 - \left.\frac{2x^3}{3}\right|_0^1$

$= 4 - 1 - \frac{2}{3} = \frac{7}{3}$

14. The sum of all values of x in $[0, 2\pi]$, for which $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$, is equal to:

(1) 11π (2) 12π (3) 8π (4) 9π

Sol. (4)

$(\sin x + \sin 4x) + (\sin 2x + \sin 3x) = 0$

$\Rightarrow 2\sin\frac{5x}{2}\left\{\cos\frac{3x}{2} + \cos\frac{x}{2}\right\} = 0$

$\Rightarrow 2\sin\frac{5x}{2}\left\{2\cos x\cos\frac{x}{2}\right\} = 0$

$2\sin\frac{5x}{2} = 0 \Rightarrow \frac{5x}{2} = 0$, p, 2π, 3π, 4π, 5π

$\Rightarrow x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$

$\cos\frac{x}{2} = 0 \Rightarrow \frac{x}{2} = \frac{x}{2} \Rightarrow x = \pi$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

So cum $= 6\pi + \pi + 2\pi = 9\pi$

15. The values of a and b, for which the system of equations

$2x + 3y + 6z = 8$

$x + 2y + az = 5$

$3x + 5y + 9z = b$

has no solution, are:

(1) $a = 3$, $b = 13$ (2) $a \neq 3$, $b \neq 13$ (3) $a \neq 3$, $b = 3$ (4) $a = 3$, $b \neq 13$

Sol. **(4)**

$$D = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 2 & a \\ 3 & 5 & 9 \end{vmatrix} = 3 - a$$

$$D = \begin{vmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \\ 3 & 5 & b \end{vmatrix} = b - 13$$

If $a = 3$, $b \neq 13$, no solution.

16. Let an ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a^2 > b^2$, passes through $\left(\sqrt{\frac{3}{2}}, 1\right)$ and has ecentricity $\frac{1}{\sqrt{3}}$. If a circle, centered at focus $F(\alpha, 0)$, $\alpha > 0$, of E and radius $\frac{2}{\sqrt{3}}$, intersects E at two points P and Q, then PQ^2 is equal to:

(1) $\frac{8}{3}$ (2) $\frac{4}{3}$ (3) 3 (4) $\frac{16}{3}$

Sol. **(4)**

$\frac{3}{2a^2} + \frac{1}{b^2} = 1$ and $1 - \frac{b^2}{a^2} = \frac{1}{3}$

$\Rightarrow a^2 = 3$ $b^2 = 2$

$\Rightarrow \frac{x^2}{3} + \frac{y^2}{2} = 1$ ...(i)

Its focus is (1, 0)

Now, eqn of circle is

$(x-1)^2 + y^2 = \frac{4}{3}$ ...(ii)

Solving (i) and (ii) we get

$y = \pm\frac{2}{\sqrt{3}}, x = 1$

$\Rightarrow PQ^2 = \left(\frac{4}{\sqrt{3}}\right)^2 = \frac{16}{3}$

17. The locus of the centroid of the triangle formed by any point P on the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, and its foci is:

(1) $9x^2 - 16y^2 + 36x + 32y - 36 = 0$ (2) $16x^2 - 9y^2 + 32x + 36y - 36 = 0$

(3) $16x^2 - 9y^2 + 32x + 36y - 144 = 0$ (4) $9x^2 - 16y^2 + 36x + 32y - 144 = 0$

Sol. **(2)**

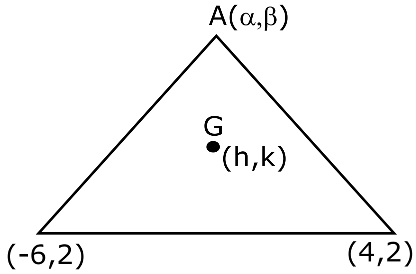
Given hyperbola is

$16(x + 1)^2 - 9(y - 2)^2 = 164 + 16 - 36 = 144$

$\Rightarrow \frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

Eccentricity, $e = \sqrt{1+\frac{16}{9}} = \frac{5}{3}$

$\Rightarrow$ foci are (4, 2) and (−6, 2)



Let the centroic be (h, k)

& $A(\alpha, \beta)$ be point on hyperbola

So $h = \frac{\alpha-6+4}{3}$, $k = \frac{\beta+2+2}{3}$

$\Rightarrow \alpha = 3h + 2, \beta = 3k - 4$

(α, β) lies on hyperbola so

$16(3h + 2 + 1)^2 - 9(3k - 4 - 2)^2 = 144$

$\Rightarrow 144(h + 1)^2 - 81(k - 2)^2 = 144$

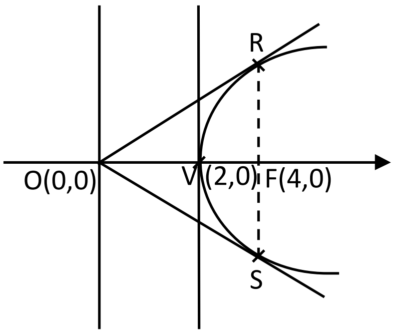
$\Rightarrow 16(h^2 + 2h + 1) - 9(k^2 - 4k + 4) = 16$

$\Rightarrow 16x^2 - 9y^2 + 32x + 36y - 36 = 0$

18. Let a parabola P be such that its vertex and focus lie on the positive x- axis at a distance 2 and 4 units from the origin, respectively. If tangents are drawn from O (0,0) to the parabola P which meet P at S and R, then the area (in sq. units) of ΔSOR is equal to:

(1) $16\sqrt{2}$ (2) 32 (3) 16 (4) $8\sqrt{2}$

Sol. (3)



Clearly Rs is latus-rectum

$\because$ VF = 2 = a

$\therefore$ RS = 4a = 8

Now OF = 2a = 4

$\Rightarrow$ Area of traingle ORS = 16

19. The value of the definite integral

$$\int_{\pi/24}^{5\pi/24} \frac{dx}{1+\sqrt[3]{\tan 2x}} \text{ is :}$$

(1) $\frac{\pi}{18}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{12}$

Sol. (4)

Let I = $\int_{\pi/24}^{5\pi/24} \frac{(\cos 2x)^{1/3}}{(\cos 2x)^{1/3}+(\sin 2x)^{1/3}} dx$ ...(i)

$$\Rightarrow I = \int_{\pi/24}^{5\pi/24} \frac{\left(\cos\left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}}{\left(\cos\left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}+\left(\sin\left\{2\left(\frac{\pi}{4}-x\right)\right\}\right)^{\frac{1}{3}}} dx$$

$$\left\{\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right\}$$

So I = $\int_{\pi/24}^{5\pi/24} \frac{(\sin 2x)^{1/3}}{(\sin 2x)^{1/3}+(\cos 2x)^{1/3}} dx$ ...(ii)

Hence 21 = $\int_{\pi/24}^{5\pi/24} dx$ [(i) + (ii)]

$$\Rightarrow 21 = \frac{4\pi}{24} \Rightarrow \boxed{I = \frac{\pi}{12}}$$

20. Let y = y(x) be the solution of the differential equation

$\frac{dy}{dx}$ = 1+ x $e^{y-x}, -\sqrt{2} < x < \sqrt{2}$, y (0) = 0 then, the minimum value of y(x), x $\in \left(-\sqrt{2}, \sqrt{2}\right)$ is equal to:

(1) $\left(1-\sqrt{3}\right)-\log_e\left(\sqrt{3}-1\right)$

(2) $\left(2+\sqrt{3}\right)+\log_e 2$

(3) $\left(2-\sqrt{3}\right)-\log_e 2$

(4) $\left(1+\sqrt{3}\right)-\log_e\left(\sqrt{3}-1\right)$

Sol. **(1)**

$$\frac{dy - dx}{e^{y-x}} = x\,dx$$

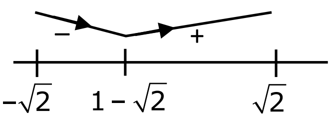
$$\Rightarrow \frac{dy - dx}{e^{y-x}} = x\,dx$$

$$\Rightarrow -e^{x-y} = \frac{x^2}{2} + c$$

At x = 0, y = 0 $\Rightarrow c = -1$

$$\Rightarrow e^{x-y} = \frac{2-x^2}{2}$$

$$\Rightarrow y = x - \ell n\left(\frac{2-x^2}{2}\right)$$



$$\Rightarrow \frac{dx}{dx} = 1 + \frac{2x}{2-x^2} = \frac{2+2x-x^2}{2-x^2}$$

So minimum value occurs at x = $1-\sqrt{3}$

$$y\left(1-\sqrt{3}\right) = \left(1-\sqrt{3}\right) - \ell n\left(\frac{2-\left(4-2\sqrt{3}\right)}{2}\right)$$

$$= \left(1-\sqrt{3}\right) - \ell n\left(\sqrt{3}-1\right)$$

SECTION B

1. Let $\vec{p} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{q} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. If a vector $\vec{r} = \left(\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}\right)$ is perpendicular to each of the vectors $\left(\vec{p}+\vec{q}\right)$ and $\left(\vec{p}-\vec{q}\right)$, and $\left|\vec{r}\right| = \sqrt{3}$, then $|\alpha| + |\beta| + |\gamma|$ is equal to ______.

Sol. **(3)**

$\vec{p} = 2\hat{i} + 3\hat{j} + \hat{k}$ (Given)

$\vec{q} = \hat{i} + 2\hat{j} + \hat{k}$

Now $(\vec{p}+\vec{q})\times(\vec{p}-\vec{q}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 2 \\ 1 & 1 & 0 \end{vmatrix}$

$= 2\hat{i}-2\hat{j}-2\hat{k}$

$\Rightarrow \vec{r} = \pm\sqrt{3}\dfrac{\left((\vec{p}+\vec{q})\times(\vec{p}-\vec{q})\right)}{\left|(\vec{p}+\vec{q})\times(\vec{p}-\vec{q})\right|} = = \pm\dfrac{\sqrt{3}\left(-2\hat{i}-2\hat{j}-2\hat{k}\right)}{\sqrt{2^2+2^2+2^2}}$

$\vec{r} = \pm\left(-\hat{i}-\hat{j}-\hat{k}\right)$

According to question

$\vec{r} = \alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$

So $|\alpha| = 1$, $|\beta| = 1$, $|\gamma| = 1$

$\Rightarrow |\alpha| + |\beta| + |\gamma| = 3$

2. The term independent of 'x' in the expansion of $\left(\dfrac{x+1}{x^{2/3}-x^{1/3}+1}-\dfrac{x-1}{x-x^{1/2}}\right)^{10}$, where x $\neq$0, 1 is equal to______.

Sol. (210)

$\left(\left(x^{1/3}+1\right)-\left(\dfrac{x^{1/2}+1}{x^{1/2}}\right)\right)^{10}$

$= \left(x^{1/3}\ \dfrac{1}{x^{1/2}}\right)^{10}$

Now General Term

$T_{r+1} = {}^{10}C_r\,(x^{1/3})^{10-r}\,.\left(-\dfrac{1}{x^{1/2}}\right)^r$

For independent term

$\dfrac{10-r}{3}-\dfrac{r}{2} = 0 \Rightarrow r = 4$

$\Rightarrow T_5 = {}^{10}C_4 = 210$

3. The ratio of the coefficient of the middle term in the expansion of $(1+x)^{20}$ and the sum of the coefficients of two middle terms in expansion of $(1+x)^{19}$ is _______.

Sol. (1)

Coeff. of middle term in $(1 + x)^{20} = {}^{20}C_{10}$

& Sum of Coeff. of two middle terms in

$(1 + x)^{19} = {}^{19}C_9 + {}^{19}C_{10}$

So required ratio $= \dfrac{{}^{20}C_{10}}{{}^{19}C_9+{}^{19}C_{10}}\ \dfrac{{}^{20}C_{10}}{{}^{19}C_9+{}^{19}C_{10}} = \dfrac{{}^{20}C_{10}}{{}^{20}C_{10}} = 1$

4. There are 5 students in class 10, 6 students in class 11 and 8 students in class 12. If the number of ways, in which 10 students can be selected from them so as to include at least 2 students from each class and at most 5 students from the total 11 students of class 10 and 11 is 100 k, then k is equal to______

Sol. **(238)**

Class	10^{th}	11^{th}	12^{th}
Total student	5	6	8
	2	3	$5 \Rightarrow {}^5C_2 \times {}^6C_3 \times {}^8C_5$
Number of selection	2	2	$6 \Rightarrow {}^5C_2 \times {}^6C_3 \times {}^8C_6$
	3	2	$5 \Rightarrow {}^5C_3 \times {}^6C_2 \times {}^8C_5$

$\Rightarrow$Total number of ways = 23800

According to question

100 K = 23800

$\Rightarrow$K = 238

5. Let M = $\left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \{\pm 3, \pm 2, \pm 1, 0\}\right\}$. Define f: M $\to$ Z, as f(A) = det (A) for all A $\in$ M, where Z is set of all integers. Then the number of A $\in$ M such that f(A) = 15 is equal to_____.

Sol. **(16)**

$|A|$ = ad – bc = 15

where a,b, c, d $\in \{\pm 3, \pm 2, \pm 1, 0\}$

Case I ad = 9 & bc = –6

For ad possible pairs are (3,3), (–3, –3)

For bc possible pairs are (3,–2), (–3,2),(–2,3),(2,–3)

So total matrix = 2 × 4 = 8

Case II ad = 6 & bc = –9

Similarly total matrix = 2 × 4 = 8

$\Rightarrow$ Total such matrices are = 16

6. Consider the following frequency distribution:

Class:	10-20	20-30	30-40	40-50	50-60
Frequency:	α	110	54	30	β

If the sum of all frequencies is 584 and median is 45, then $|\alpha - \beta|$ is equal to _______

Sol. **(164)**

Class	**Frequency**	**C.F.**
10-20	α	α
20-30	110	α + 110
30-40	54	α + 164
40-50	30	α + 194
50-60	β	α + b + 194 = 584

$N = \sum f = 584$

$\alpha + \beta = 390$

$$\text{Median }(m) = \ell + \left[\frac{\left(\frac{N}{2}\right) - c}{f}\right] \times h$$

$$N = \frac{584}{2} = 292$$

$$m = 45 = 40 + \left[\frac{292 - (\alpha + 164)}{30}\right] \times 10$$

$$45 = 40 + \left(\frac{128 - \alpha}{3}\right)$$

$15 = 128 - \alpha$
$\alpha = 113$
$\beta = 277$
$|\alpha - \beta| = |113 - 277| = 164$

7. Let y = y(x) be solution of the following differential equation $e^y \frac{dy}{dx} - 2e^y \sin x + \sin x \cos^2 x = 0$, $y\left(\frac{\pi}{2}\right) = 0$. If $y(0) = \log_e(\alpha + \beta e^{-2})$, then $4(\alpha + \beta)$ is equal to____.

Sol. **(4)**
Let $e^y = t$

$$\Rightarrow \frac{dt}{dx} - (2\sin x)t = -\sin x \cos^2 x$$

I.F. $= e^{2\cos x}$

$$\Rightarrow t \cdot e^{2\cos x} = \int e^{2\cos x} \cdot (-\sin x \cos^2 x)\,dx$$

$$\Rightarrow e^y \cdot e^{2\cos x} = \int e^{2z} \cdot z^2 dz, z = e^{2\cos x}$$

$$\Rightarrow e^y \cdot e^{2\cos x} = \frac{1}{2}.\cos^2 x.e^{2\cos x} + \frac{e^{2\cos x}}{4} + C$$

at $x = \frac{\pi}{2}$, $y = 0 \Rightarrow C = \frac{3}{4}$

$$\Rightarrow e^y = \frac{1}{2}\cos^2 x - \frac{1}{2}\cos x + \frac{1}{4} + \frac{3}{4}.e^{-2\cos x}$$

$$\Rightarrow y = \log\left[\frac{\cos^2 x}{2} - \frac{\cos x}{2} + \frac{1}{4} + \frac{3}{4}e^{-2\cos x}\right]$$

Put x = 0

$$\Rightarrow y = \log\left[\frac{1}{4} + \frac{3}{4}e^{-2\cos x}\right] \Rightarrow \alpha = \frac{1}{4}, \beta = \frac{3}{4}$$

8. If α, β are roots of the equation $x^2 + 5\sqrt{2}x + 10 = 0$, $\alpha > \beta$ and $P_n = \alpha^n - \beta^n$ for each positive integer n, then the value of $\left(\frac{P_{17}P_{20} + 5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2}\right)$ is equal to ________.

Sol. **(1)**
$x^2 + 5\sqrt{2}x + 10 = 0$
& $p_n = \alpha^n - \beta^n$ (Given)

Now $\frac{P_{17}P_{20}+5\sqrt{2}p_{17}P_{19}}{P_{18}P_{19}+5\sqrt{2}P_{18}^2} = \frac{P_{17}\left(P_{20}5\sqrt{2}P_{19}\right)}{P_{18}\left(P_{19}+5\sqrt{2}P_{18}\right)}$

$$\frac{P_{17}\left(\alpha^{20}-\beta^{20}+5\sqrt{2}\left(\alpha^{19}-\beta^{19}\right)\right)}{P_{18}\left(\alpha^{19}-\beta^{19}+5\sqrt{2}\left(\alpha^{18}-\beta^{18}\right)\right)}$$

$$\frac{P_{17}\left(\alpha^{19}\left(\alpha+5\sqrt{2}\right)-\beta^{19}\left(\beta+5\sqrt{2}\right)\right)}{P_{18}\left(\alpha^{18}\left(\alpha+5\sqrt{2}\right)-\beta^{18}\left(\beta+5\sqrt{2}\right)\right)}$$

Since $\alpha+5\sqrt{2} = -10/\alpha$ ....(1)

$\&\beta+5\sqrt{2} = -10/\beta$ ....(2)

Now put there values in above expression

$= -\frac{10P_{17}P_{18}}{-10P_{18}P_{17}} = 1$

9. Let $S=\left\{n\in N\left|\begin{pmatrix}0 & i\\1 & 0\end{pmatrix}^n\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}\forall a,b,c,d\in R\right.\right\}$, where $i=\sqrt{-1}$. Then the number of 2- digit numbers in the set S is ______.

Sol. **(11)**

Lex X = $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ & A = $\begin{pmatrix}o & i\\1 & 0\end{pmatrix}^n$

$\Rightarrow$ AX = IX

$\Rightarrow$A = I

$\Rightarrow\begin{pmatrix}0 & i\\1 & 0\end{pmatrix}^n = I$

$\Rightarrow A^8 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\Rightarrow$n is multiple of 8

So number of 2 digit numbers in the set

S= 11 (16, 24, 32, ..., 96)

10. If the value of $\left(1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+....upto\,\infty\right)^{\log_{(0.25)}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+....upto\,\infty\right)}$ is l, then l^2 is equal to _____.

Sol. **(3)**

$$\ell = \left(\underbrace{1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}}_{s}+...\right)^{\log_{(0.25)}\left(\frac{1}{3}+\frac{1}{3^2}+....\right)}$$

$S = 1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+....$

$\frac{S}{3}=\frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+....$

– – – –

$$\frac{2S}{3} = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + ...$$

$$\frac{2S}{3} = \frac{4}{3} + \frac{4}{3^2} + \frac{4}{3^3} + ...$$

$$S = \frac{3}{2}\left(\frac{4/3}{1-1/3}\right) = 3$$

Now, $\ell = (3)^{\log_{0.25}\left(\frac{1/3}{1-1/3}\right)}$

$$\ell = 3^{\log_{((1/4))}\left(\frac{1}{2}\right)} = 3^{1/2} = \sqrt{3}$$

$\Rightarrow \ell^2 = 3$