JEE MAIN 4^{th} Attempt

MATHEMATICS

27^{th} August 2021 [SHIFT – 2]

QUESTION WITH SOLUTION


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SECTION – A

1. Let A(a,0), B(b,2b+1) and C(0, b), $b \neq 0$, $|b| \neq 1$ be points such that the area of triangle ABC is 1sq. unit, then the sum of all possible values of a is:

(1) $\frac{-2b^2}{b+1}$

(2) $\frac{2b^2}{b+1}$

(3) $\frac{-2b}{b+1}$

(4) $\frac{2b}{b+1}$

Ans. (3)

Sol. $$\left|\frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ b & 2b+1 & 1 \\ 0 & b & 1 \end{vmatrix}\right| = 1$$

$$\Rightarrow \begin{vmatrix} a & 0 & 1 \\ b & 2b+1 & 1 \\ 0 & b & 1 \end{vmatrix} = \pm 2$$

$$\Rightarrow a(2b+1-b)-0+1\left(b^2-0\right)=\pm 2$$

$$\Rightarrow a=\frac{\pm 2-b^2}{b+1}$$

$$\therefore a=\frac{2-b^2}{b+1} \text{ and } a=\frac{-2-b^2}{b+1}$$

sum of possible values of ‘a’ is

$$=\frac{-2b^2}{a+1}$$

2. If $0 < x < 1$ and $y = \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \ldots$, then the value of e^{1+y} at $x = \frac{1}{2}$ is:

(1) $\frac{1}{2}e^2$

(2) 2e

(3) $2e^2$

(4) $\frac{1}{2}\sqrt{e}$

Ans. **(1)**

Sol. $y = \left(1 - \frac{1}{2}\right)x^2 + \left(1 - \frac{1}{3}\right)x^3 + \ldots$

$= (x^2 + x^3 + x^4 + \ldots) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right)$

$= \frac{x^2}{1-x} + x - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)$

$= \frac{x}{1-x} + \ell n(1-x)$

$x = \frac{1}{2} \Rightarrow y = 1 - \ell n 2$

$e^{1+y} = e^{1+1-\ell n 2}$

$= e^{2-\ell n 2} = \frac{e^2}{2}$

3. The equation of the plane passing through the line of intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$ and parallel to the x-axis is :

(1) $\vec{r}.(\hat{i}+3\hat{k})+6=0$

(2) $\vec{r}.(\hat{j}-3\hat{k})-6=0$

(3) $\vec{r}.(\hat{i}-3\hat{k})+6=0$

(4) $\vec{r}.(\hat{j}-3\hat{k})+6=0$

Ans. **(4)**

Equation of planes are

$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-1=0 \Rightarrow x+y+z-1=0$

and $\vec{r}\times(2\hat{i}+3\hat{j}-\hat{k})+4=0 \Rightarrow 2x+3y-z+4=0$

equation of planes through line of intersection of these plane is :–

$(x+y+z-1)+\lambda(2x+3y-z+4)=0$

$\Rightarrow (1+2\lambda)x+(1+3\lambda)y+(1-\lambda)z-1+4\lambda=0$

But this plane is parallel to x–axis whose direction are (1,0,0)

$\therefore (1+2\lambda)1+(1+3\lambda)0+(1-\lambda)0=0$

$\lambda=-\frac{1}{2}$

$\therefore$ Required plane is

$0x+\left(1-\frac{3}{2}\right)y+\left(1+\frac{1}{2}\right)z-1+4\left(\frac{-1}{2}\right)=0$

$\Rightarrow \frac{-y}{2}+\frac{3}{2}z-3=0$

$\Rightarrow y-3z+6=0$

$\Rightarrow \vec{r}.(\hat{j}-3\hat{k})+6=0$

4. Each of the persons A and B independently tosses three fair coins. The probability that both of them get the same number of heads is:

(1) $\frac{5}{8}$

(2) $\frac{5}{16}$

(3) $\frac{1}{8}$

(4) 1

Ans. (2)

Sol. C–I 	'0' Head

T T T $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{1}{64}$

C–II 	'1' Head

H T T $\left(\frac{3}{8}\right)\left(\frac{3}{8}\right) = \frac{9}{64}$

C–III 	'2' Heads

H H T $\left(\frac{3}{8}\right)\left(\frac{3}{8}\right) = \frac{9}{64}$

C–IV 	'3' Heads

H H H $\left(\frac{1}{8}\right)\left(\frac{1}{8}\right) = \frac{1}{64}$

Total probability $= \frac{5}{16}$

5. Let $[\lambda]$ be the greatest integer less than or equal to λ. The set of all value of λ for which the system of linear equations x + y + z= 4, 3x+2y+5z = 3, 9x+4y+(28+$[\lambda]$)z = $[\lambda]$ has a solution is:

(1) $(-\infty,-9)\cup[-8,\infty)$

(2) $(-\infty,-9)\cup(-9,\infty)$

(3) $[-9,-8]$

(4) R

Ans. **(1)**

Sol. $D=\begin{vmatrix}1 & 1 & 1\\ 3 & 2 & 5\\ 9 & 4 & 28+[\lambda]\end{vmatrix}=-24-[\lambda]+15=-[\lambda]-9$

If $\left[\lambda\right]+9\neq 0$ then unique solution

if $\left[\lambda\right]+9=0$ then $D_1 = D_2 = D_3 = 0$

so infinite solutions

Hence λ can be any real number.

6. A box open from top is made from a rectangular sheet of dimension a × b by cutting squares each of side x from each of the four corners fand folding up the flaps. If the volume of the box is maximum, then x is equal to :

(1) $\dfrac{a+b-\sqrt{a^2+b^2-ab}}{6}$

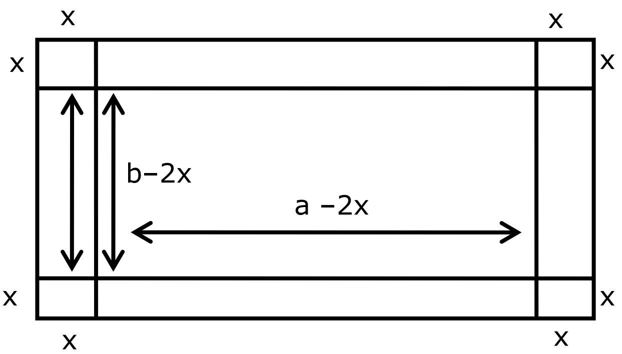
(2) $\dfrac{a+b+\sqrt{a^2+b^2-ab}}{6}$

(3) $\dfrac{a+b-\sqrt{a^2+b^2-ab}}{12}$

(4) $\dfrac{a+b-\sqrt{a^2+b^2+ab}}{6}$

And. **(4)**

Sol.



$V = \ell\ .\ b.\ h = (a-2x)\ (b-2x)\ x$

$\Rightarrow V(x)=(2x-a)(2x-b)x$

$\Rightarrow V(x)=4x^3-2(a+b)x^2+abx$

$\Rightarrow \frac{d}{dx}V(x)=12x^2-4(a+b)x+ab$

$\frac{d}{dx}(V(x))=0 \Rightarrow 12x^2-4(a+b)x+ab=0<^{\alpha}_{\beta}$

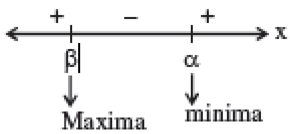
$\Rightarrow x=\frac{4(a+b)\pm\sqrt{16(a+b)^2-48ab}}{2(12)}$

$=\frac{(a+b)\pm\sqrt{a^2+b^2-ab}}{6}$

Let $x = \alpha = \frac{(a+b)+\sqrt{a^2+b^2-ab}}{6}$

$\beta=\frac{(a+b)-\sqrt{a^2+b^2-ab}}{6}$

Now, $12(x-\alpha)(x-\beta)=0$



$\therefore x = \beta$

$=\frac{a+b-\sqrt{a^2+b^2-ab}}{b}$

7. Let $A=\begin{pmatrix} [x+1] & [x+2] & [x+3] \\ [x] & [x+3] & [x+3] \\ [x] & [x+2] & [x+4] \end{pmatrix}$, where [t] denotes the greatest integer less than or equal to t. If det(A) = 192, then the set of value of x is the interval:

(1) [62,63)

(2) [60,61)

(3) [68,69)

(4) [65,66)

Ans. (1)

Sol. $\begin{vmatrix} [x+1] & [x+2] & [x+3] \\ [x] & [x+3] & [x+3] \\ [x] & [x+2] & [x+4] \end{vmatrix} = 192$

$R_1 \to R_1 - R_3 \ \& \ R_2 \to R_2 - R_3$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ [x] & [x]+2 & [x]+4 \end{bmatrix} = 192$

2[x] + 6 + [x] = 192 ⇒[x] = 62

8. If $\lim_{x\to\infty}\left(\sqrt{x^2-x+1}-ax\right)=b$, then the ordered pair (a, b) is :

(1) $\left(1, \frac{1}{2}\right)$

(2) $\left(-1, \frac{1}{2}\right)$

(3) $\left(-1, -\frac{1}{2}\right)$

(4) $\left(1, -\frac{1}{2}\right)$

Ans. (4)

Sol. $\lim_{x\to\infty}\left(\sqrt{x^2-x+1}\right)-ax=b \quad (\infty-\infty)\text{form}$

$\Rightarrow a>0$

Now, $\lim_{x\to\infty}\frac{\left(x^2-x+1-a^2x^2\right)}{\sqrt{x^2-x+1}+ax}=b$

$\Rightarrow \lim_{x\to\infty}\frac{\left(1-a^2\right)x^2-x+1}{\sqrt{x^2-x+1}+ax}=b$

$\Rightarrow \lim_{x\to\infty}\frac{\left(1-a^2\right)x^2-x+1}{x\left(\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a\right)}=b$

$\Rightarrow 1-a^2=0\Rightarrow a=1$

Now, $\lim_{x\to\infty}\frac{-x+1}{x\left(\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a\right)}=b$

$\Rightarrow \frac{-1}{1+a}=b\Rightarrow b=-\frac{1}{2}$

$(a,b)=\left(1,-\frac{1}{2}\right)$

9. The value of the integral $\int_0^1\frac{\sqrt{x}dx}{(1+x)(1+3x)(3+x)}$ is:

(1) $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$

(2) $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{6}\right)$

(3) $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{2}\right)$

(4) $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{6}\right)$

Ans. **(1)**

Sol. $I=\int_0^1 \frac{\sqrt{x}}{(1+x)(1+3x)(3+x)}dx$

Let $x = t^2 \Rightarrow dx = 2t.\ dt$

$$I=\int_0^1 \frac{t(2t)}{\left(t^2+1\right)\left(1+3t^2\right)\left(3+t^2\right)}dt$$

$$I=\int_0^1 \frac{\left(3t^2+1\right)-\left(t^2+1\right)}{\left(3t^2+1\right)\left(t^2+1\right)\left(3+t^2\right)}dt$$

$$I=\int_0^1 \frac{dt}{\left(t^2+1\right)\left(3+t^2\right)}-\int_0^1 \frac{dt}{\left(1+3t^2\right)\left(3+t^2\right)}$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{1+t^2}-\frac{1}{2}\int_0^1 \frac{dt}{t^2+3}+\frac{1}{8}\int_0^1 \frac{dt}{t^2+3}-\frac{3}{8}\int_0^1 \frac{dt}{\left(1+3t^2\right)}$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{t^2+1}-\frac{3}{8}\int_0^1 \frac{dt}{t^2+3}-\frac{3}{8}\int_0^1 \frac{dt}{1+3t^2}$$

$$=\frac{1}{2}\left(\tan^{-1}(t)\right)_0^1-\frac{3}{8\sqrt{3}}\left(\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right)_0^1-\frac{3}{8\sqrt{3}}\left(\tan^{-1}(\sqrt{3}t)\right)_0^1$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)-\frac{\sqrt{3}}{8}\left(\frac{\pi}{6}\right)-\frac{\sqrt{3}}{8}\left(\frac{\pi}{3}\right)$$

$$=\frac{\pi}{8}-\frac{\sqrt{3}}{16}\pi$$

$$=\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$$

10. If $y(x) = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), x \in \left(\frac{\pi}{2}, \pi\right)$, then $\frac{dy}{dx}$ at $x=\frac{5\pi}{6}$ is:

(1) 0

(2) −1

(3) $\frac{1}{2}$

(4) $-\frac{1}{2}$

Ans. (4)

Sol. $y(x)=\cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right]$

$y(x)=\cot^{-1}\left(\tan\frac{x}{2}\right)=\frac{\pi}{2}-\frac{x}{2}$

$y'(x)=\frac{-1}{2}$

11. Two poles, AB of length a metres and CD of length a + b (b $\neq$ a)metres are erected at the same horizontal level with bases at B and D. If BD = X and $\tan\angle ACB=\frac{1}{2}$, then:

(1) $x^2 + 2(a + 2b)x + a(a + b) = 0$

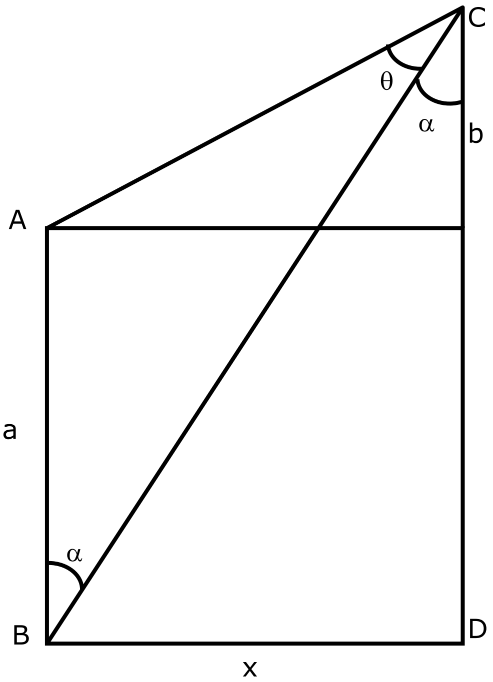
(2) $x^2 - 2ax + b(a + b) = 0$

(3) $x^2 + 2(a + 2b)x - b(a + b) = 0$

(4) $x^2 - 2ax + a(a + b) = 0$

Ans. (2)

Sol.



$\tan\theta=\frac{1}{2}$

$\tan(\theta+\alpha)=\frac{x}{b},\tan\alpha=\frac{x}{a+b}$

$\Rightarrow \frac{1}{2}+\frac{x}{a+b}$

$\Rightarrow \frac{\frac{1}{2}+\frac{x}{a+b}}{1-\frac{1}{2}\times\frac{x}{a+b}}=\frac{x}{b}$

$\Rightarrow x^2-2ax+ab+b^2=0$

12. If the solution curve of the differential equation $(2x - 10y^3)dy + ydx = 0$, passes through the points (0,1) and $(2,\beta)$, then β is a root of the equation :

(1) $2y^5 - 2y - 1 = 0$

(2) $y^5 - y^2 - 1 = 0$

(3) $y^5 - 2y - 2 = 0$

(4) $2y^5 - y^2 - 2 = 0$

Ans. **(2)**

Sol. $\left(2x\text{-}10y^3\right)dy+ydx=0$

$$\Rightarrow \frac{dx}{dy}+\left(\frac{2}{y}\right)x=10y^2$$

$$\text{I.F.} = e^{\int \frac{2}{y}dy}=e^{2\ln(y)}=y^2$$

Solution of D.E. is

$$\therefore x.\, y = \int \left(10y^2\right) y^2 \cdot dy$$

$$xy^2 = \frac{10y^5}{5} + C \Rightarrow xy^2 = 2y^5 + C$$

It passes through $(0,1) \rightarrow 0 = 2 + C \Rightarrow C = -2$

$\therefore$ Curve is $\boxed{xy^2 = 2y^5 - 2}$

Now, it passes through $(2,\beta)$

$$2\beta^2=2\beta^5\text{-}2 \Rightarrow \beta^5\text{-}\beta^2\text{-}1=0$$

$\therefore \beta$ is root of an equation $\boxed{y^5 - y^2 - 1 = 0}$

13. Let $\mathbb{Z}$ be the set of all integers,

$A=\{(x,y)\in\mathbb{Z}\times\mathbb{Z}:(x-2)^2+y^2\le 4\}$

$B=\{(x,y)\in\mathbb{Z}\times\mathbb{Z}:x^2+y^2\le 4\}$ and

$C=\{(x,y)\in\mathbb{Z}\times\mathbb{Z}:(x-2)^2+(y-2)^2\le 4\}$

If the total number of relation from $A\cap B$ to $A\cap c$ is 2^P, then the value of p is :

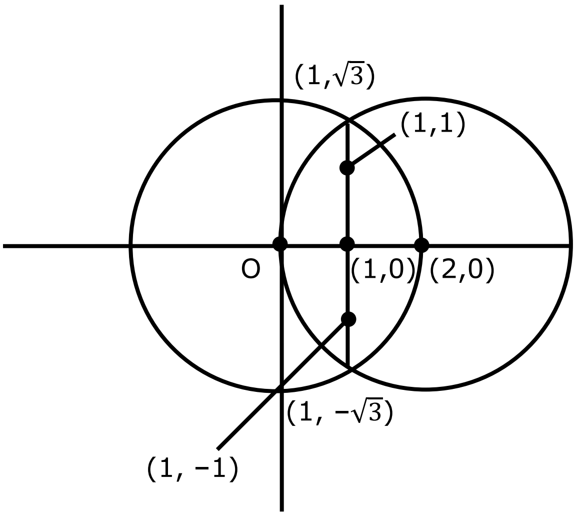
(1) 16

(2) 25

(3) 49

(4) 9

Ans. (2)



Sol.

$(x-2)^2+y^2\le 4$

$x^2+y^2\le 4$

No. of points common in C_1 & C_2 is 5.
(0, 0) , (1, 0) , (2, 0) , (1, 1), (1, −1)
Similarly in C_2 & C_3 is 5.
No. of relations = $2^{5\times5}=2^{25}$.

14. The area of the region bounded by the parabola $(y-2)^2 = (x-1)$, the tangent to it at the point whose ordinate is 3 and the x–axis is :

(1) 6

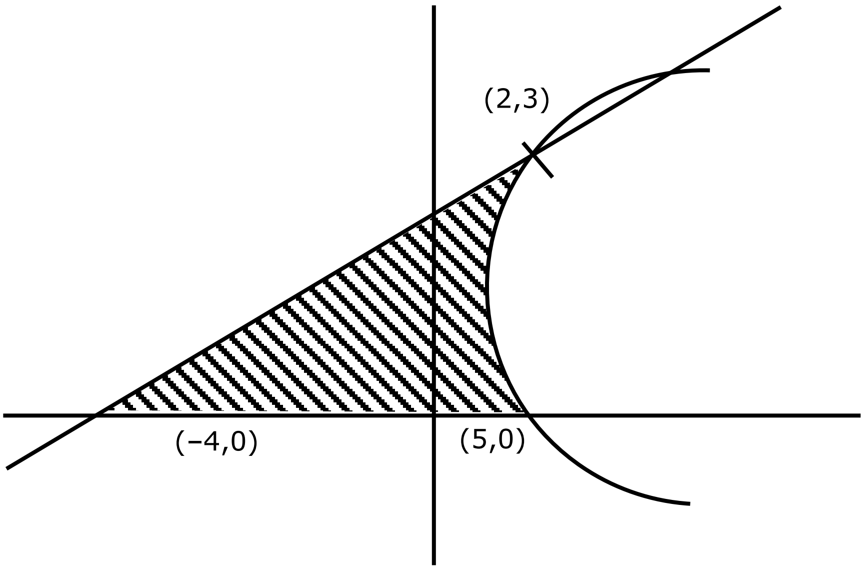
(2) 4

(3) 10

(4) 9

Ans. (4)

Sol.



$y = 3 \Rightarrow x = 2$

Point is (2,3)

Diff. w.r.t x

$2(y-2)y' = 1$

$\Rightarrow y' = \frac{1}{2(y-2)}$

$\Rightarrow y'_{(2,3)} = \frac{1}{2}$

$\Rightarrow \frac{y-3}{x-2} = \frac{1}{2} \Rightarrow x-2y+4 = 0$

Area $= \int_0^3 \left((y-2)^2+1-(2y-4)\right)dy$

= 9 sq. units

15. Let M and m respectively be the maximum and minimum values of the function f(x) = $\tan^{-1}$(sinx + cosx) in $\left[0,\frac{\pi}{2}\right]$ Then the value of tan(M–m) is equal to :

(1) $3-2\sqrt{2}$

(2) $3+2\sqrt{2}$

(3) $2-\sqrt{3}$

(4) $2+\sqrt{3}$

Ans. (1)

Sol.

Let g(x) = sinx + cosx $=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

$g(x) \in [1,\sqrt{2}]$ for $x \in [0,\pi/2]$

$f(x)=\tan^{-1}(\sin x+\cos x) \in \left[\frac{\pi}{4},\tan^{-1}\sqrt{2}\right]$

$\tan\left(\tan^{-1}\sqrt{2}-\frac{\pi}{4}\right) = \frac{\sqrt{2}-1}{1+\sqrt{2}} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = 3-2\sqrt{2}$

16. A differential equation representing the family of parabolas with axis parallel to y–axis and whose length of latus rectum is the distance of the point (2,–3) from the line 3x+4y = 5 is given by :

(1) $10\frac{d^2y}{dx^2}=11$

(2) $11\frac{d^2y}{dx^2}=10$

(3) $10\frac{d^2x}{dy^2}=11$

(4) $11\frac{d^2x}{dy^2}=10$

Ans. **(2)**

Sol. $\alpha.R = \frac{|3(2)+4(-3)-5|}{5} = \frac{11}{5}$

$(x-h)^2 = \frac{11}{5}(y-k)$

Differentiate w.r.t 'x' :–

$2(x-h) = \frac{11}{5}\frac{dy}{dx}$

Again differentiate

$2 = \frac{11}{5}\frac{d^2y}{dx^2}$

$\frac{11d^2y}{dx^2} = 10$

17. The Boolean expression $(p \wedge q) \Rightarrow ((r \wedge q) \wedge p)$ is equivalent to :

(1) $(p \wedge r) \Rightarrow (p \wedge q)$

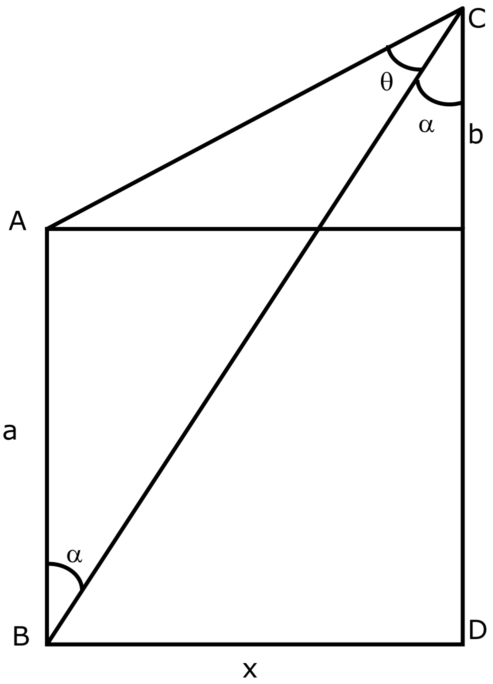
(2) $(q \wedge r) \Rightarrow (p \wedge q)$

(3) $(p \wedge q) \Rightarrow (r \wedge q)$

(4) $(p \wedge q) \Rightarrow (r \vee q)$

Ans. (1)

Sol.



$(p \wedge q) \Rightarrow ((r \wedge q) \wedge p)$

$\sim (p \wedge q) \vee ((r \wedge q) \wedge p)$

$\sim (p \wedge q) \vee ((r \wedge p) \wedge (p \wedge q)$

$\Rightarrow [\sim (p \wedge q) \vee (p \wedge q)] \wedge (\sim (p \wedge q) \vee (r \wedge p))$

$\Rightarrow t \wedge [\sim (p \wedge q) \vee (r \wedge p)]$

$\Rightarrow \sim (p \wedge q) \vee (r \wedge p)$

$\Rightarrow (p \wedge q) \Rightarrow (r \wedge p)$

18. The set of all value of $k>-1$, for which the equation $(3x^2+4x+3)^2-(k+1)(3x^2+4x+3)(3x^2+4x+2)+k(3x^2+4x+2)^2 = 0$ has real roots, is:

(1) $\left[-\frac{1}{2},1\right)$

(2) $[2,3)$

(3) $\left(1,\frac{5}{2}\right]$

(4) $\left(\frac{1}{2},\frac{3}{2}\right]-\{1\}$

Ans. **(3)**

Sol. $(3x^2 + 4x + 3)^2 - (k + 1)(3x^2 + 4x + 3)(3x^2 + 4x + 2)$
$+ k(3x^2 + 4x + 2)^2 = 0$
Let $3x^2 + 4x + 3 = a$
and $3x^2 + 4x + 2 = b \Rightarrow b = a - 1$
Given equation becomes
$\Rightarrow a^2 - (k+1)ab + kb^2 = 0$
$\Rightarrow a(a - kb) - b(a - kb) = 0$
$\Rightarrow (a - kb)(a - b) = 0 \Rightarrow a = kb$ or $a = b$ (reject)
$\because a = kb$
$\Rightarrow 3x^2 + 4x + 3 = k(3x^2 + 4x + 2)$
$\Rightarrow 3(k-1)x^2 + 4(k-1)x + (2k - 3) = 0$
for real roots
$D \geq 0$
$\Rightarrow 16(k-1)^2 - 4(3(k-1))(2k - 3) \geq 0$
$\Rightarrow 4(k-1)\{4(k-1) - 3(2k - 3)\} \geq 0$
$\Rightarrow 4(k-1)\{-2k + 5\} \geq 0$
$\Rightarrow -4(k-1)\{2k - 5\} \geq 0$
$\Rightarrow (k - 1)(2k - 5) \leq 0$

$\therefore k \in \left[1,\frac{5}{2}\right]$

$\because k \neq 1$

$\therefore k \in \left(1,\frac{5}{2}\right]$

19. The angle between the straight lines, whose direction cosines are given by the equation $2\ell+2m-n=0$ and $mn+n\ell+\ell m=0$, is :

(1) $\frac{\pi}{3}$

(2) $\pi-\cos^{-1}\left(\frac{4}{9}\right)$

(3) $\cos^{-1}\left(\frac{8}{9}\right)$

(4) $\frac{\pi}{2}$

Ans. **(4)**

Sol. $n=2(\ell+m)$

$\ell m+n(\ell+m)=0$

$\ell m+2(\ell+m)^2=0$

$2\ell^2+2m^2+5m\ell=0$

$2\left(\frac{\ell}{m}\right)^2+2+5\left(\frac{\ell}{m}\right)=0$

$2t^2+5t+2=0$

$(t+2)(2t+1)=0$

$\Rightarrow t=-2;-\frac{1}{2}$

(i) $\frac{\ell}{m}=-2$

$\frac{n}{m}=-2$

$(-2m,m,-2m)$

$(-2,1,-2)$

(ii) $\frac{\ell}{m}=-\frac{1}{2}$

$n = -2\ell$

$(\ell, -2\ell, -2\ell)$

$(1, -2, -2)$

$$\cos\theta = \frac{-2-2+4}{\sqrt{9}\sqrt{9}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

20. If two tangents drawn from a point P to the parabola $y^2 = 16(x-3)$ are at right angles, then the locus of point P is :

(1) $x + 3 = 0$

(2) $x + 2 = 0$

(3) $x + 4 = 0$

(4) $x + 1 = 0$

Ans. **(4)**

Sol. Locus is directrix of parabola
$x - 3 + 4 = 0 \Rightarrow x + 1 = 0$.

Section B

1. The probability distribution of random variable X is given by :

X	1	2	3	4	5
P(X)	K	2K	2K	3K	K

Let $p = P(1 < x < 4 | x < 3)$. If $5P = \lambda K$, then λ is equal to____________.

Ans. **(30)**

Sol. $\sum P(X) = 1 \Rightarrow k + 2k + 2k + 3k + k = 1$

$\Rightarrow k = \frac{1}{9}$

Now, $p = P\left(\frac{kX<4}{X<3}\right) = \frac{P(X=2)}{P(X<3)} = \frac{\frac{2k}{9k}}{\frac{k}{9k}+\frac{2k}{9k}} = \frac{2}{3}$

$\Rightarrow p = \frac{2}{3}$

Now, 5p = λk

$\Rightarrow (5)\left(\frac{2}{3}\right) = \lambda(1/9)$

$\Rightarrow \lambda = 30$

2. An online exam is attempted by 50 candidates out of which 20 are boys. The average marks obtained by boys is 12 with a variance 2. The variance of marks obtained by 30 girls is also 2. The average marks of all 50 candidates is 15. If μ is the average marks of girls and σ^2 is the variance of marks of 50 candidates, then $\mu + \sigma^2$ is equal to__________.

Ans. **(25)**

Sol. $\sigma_b^2 = 2$ (variance of boys) n_1 = no. of boys

$\bar{x}_b = 12$ n_2 = no. of girls

$\sigma_g^2 = 2$

$\bar{x}_g = \frac{50\times15-12\times\sigma_b}{30} = \frac{750-12\times20}{30} = 17 = \mu$

Variance of combined series

$$\sigma^2 = \frac{n_1\sigma_b^2 + n_2\sigma_g^2}{n_1+n_2} + \frac{n_1.n_2}{(n_1+n_2)^2}(\bar{x}_b - \bar{x}_g)^2$$

$$\sigma^2 = \frac{20\times2+30\times2}{20+30} + \frac{20\times30}{(20+30)^2}(12-17)^2$$

$\sigma^2 = 8$

$\Rightarrow \mu + \sigma^2 = 17 + 8 = 25$

3. Let S = {1, 2, 3, 4, 5, 6, 9}.Then the number of elements in the set $T = \{A \subseteq S : A \neq \phi$ and the sum of all the elements of A is not a multiple of 3} is____________.

Ans. **(80)**

Sol. 3n type $\rightarrow$ 3, 6, 9 = P
3n – 1 type $\rightarrow$ 2, 5 = Q
3n– 2 type $\rightarrow$ 1,4 = R
number of subset of S containing one element
which are not divisible by 3 = ${}^2C_1 + {}^2C_1 = 4$
number of subset of S containing two numbers
whose sum is not divisible by 3
$= {}^3C_1 \times {}^2C_1 + {}^3C_1 \times {}^2C_1 + {}^2C_2 + {}^2C_2 = 14$
number of subsets containing 3 elements whose
sum is not divisible by 3
$= {}^3C_2 \times {}^4C_1 + \left({}^2C_2 \times {}^2C_1\right) \times 2 + {}^3C_1\left({}^2C_2 + {}^2C_2\right) = 22$
number of subsets containing 4 elements whose
sum is not divisible by 3
$= {}^3C_3 \times {}^4C_1 + {}^3C_2\left({}^2C_2 + {}^2C_2\right) + \left({}^3C_1 {}^2C_1 \times {}^2C_2\right) \times 2$
= 4 + 6 + 12 = 22.
number of subsets of S containing 5 elements
whose sum is not divisible by 3.
${}^3C_3\left({}^2C_2 + {}^2C_2\right) + \left({}^3C_2 \times {}^2C_1 \times {}^2C_2\right) \times 2$
= 2 + 12 = 14
number of subsets of S containing 6 elements
whose sum is not divisible by 3 = 4
$\Rightarrow$ Total subsets of Set A whose sum of digits is
not divisible by 3 = 4 + 14 + 22 + 22 + 14 + 4 = 80.

4. Let $A(\sec\theta, 2\tan\theta)$ and $B(\sec\phi, 2\tan\phi)$, where $\theta + \phi = \pi/2$, be two points on the hyperbola $2x^2 - y^2 = 2$. If (α, β) is the point of the intersection of the normals to the hyperbola at A and B, then $(2\beta)^2$ is equal to____________.

Ans. **(36)**

Sol. Since, point $A(\sec\theta, 2\tan\theta)$

lies on the hyperbola

$2x^2 - y^2 = 2$

Therefore, $2\sec^2\theta - 4\tan^2\theta = 2$

$\Rightarrow 2 + 2\tan^2\theta - 4\tan^2\theta = 2$

$\Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

Similarly, for point B, we will get $\phi = 0$.

but according to question $\theta + \phi = \frac{\pi}{2}$

which is not possible.

Hence it must be a 'BONUS'

5. Let S be the sum of all solutions (in radians) of the equation $\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta = 0$ in $[0, 4\pi]$. The $\frac{8S}{\pi}$ is equal to__________.

Ans. **(56)**

Sol. Given equation

$\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta = 0$

$\Rightarrow 1 - \sin^2\theta\cos^2\theta - \sin\theta\cos\theta = 0$

$\Rightarrow 2 - (\sin 2\theta)^2 - \sin 2\theta = 0$

$\Rightarrow (\sin 2\theta)^2 + (\sin 2\theta) - 2 = 0$

$\Rightarrow (\sin 2\theta + 2)(\sin 2\theta - 1) = 0$

$\Rightarrow \sin 2\theta = 1$ or $\frac{\boxed{\sin 2\theta = -2}}{(\text{ not possible })}$

$\Rightarrow \quad 2\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

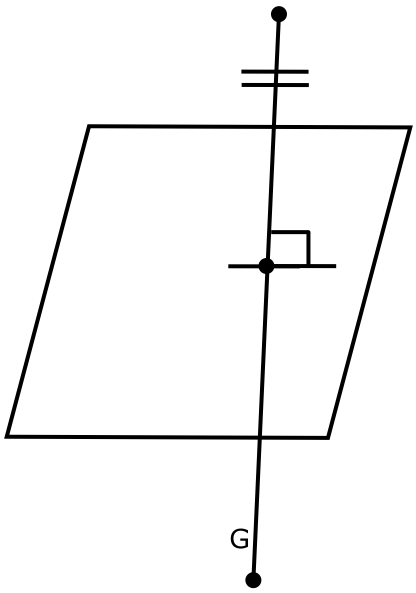
$$\Rightarrow S=\frac{\pi}{4}+\frac{5\pi}{4}+\frac{9\pi}{4}+\frac{13\pi}{4}=7\pi$$

$$\Rightarrow \frac{8S}{\pi}=\frac{8\times 7\pi}{\pi}=56.00$$

6. Let S be the mirror image of the point Q(1, 3, 4) with respect to the plane $2x-y+z+3=0$ and let R(3,5, γ) be a point of this plane. Then the square of the length of the line segment SR is ________.

Ans. **(72)**

Sol.



Since R (3,5, γ) lies on the plane $2x - y + z + 3 = 0$.

Therefore, $6 - 5 + \gamma + 3 = 0$

$\Rightarrow \gamma = -4$

Now,

dr's of line QS

are 2, −1,1

equation of line QS is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=\lambda \text{ (say)}$$

$\Rightarrow F(2\lambda+1,-\lambda+3,\lambda+4)$

F lies in the plane

$\Rightarrow 2(2\lambda+1)-(-\lambda+3)+(\lambda+4)+3=0$

$\Rightarrow 4\lambda+2+\lambda-3+\lambda+7=0$

$\Rightarrow 6\lambda+6=0 \Rightarrow \lambda=-1$

$\Rightarrow F(-1,4,3)$

Since, F is mid-point of QS.
Therefore, co-ordinated of S are (−3,5,2).

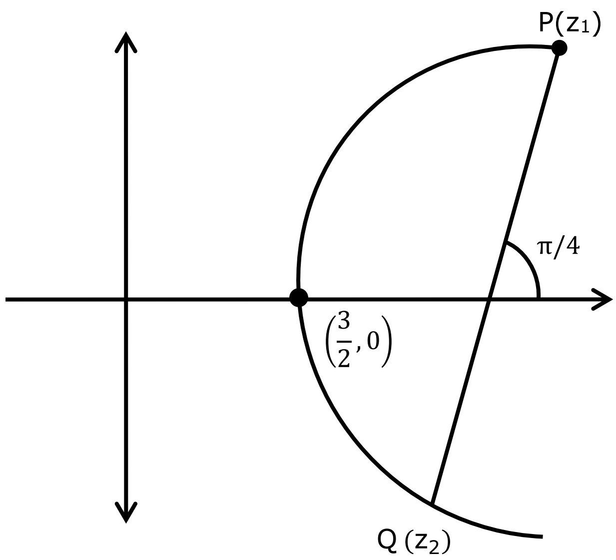
So, $SR=\sqrt{36+0+36}=\sqrt{72}$

$SR^2 = 72$

7. Let z_1 and z_2 be two complex numbers such that arg $(z_1-z_2)=\frac{\pi}{4}$ and z_1, z_2 satisfy the equation $|z-3| = Re(z)$. Then the imaginary part of z_1+z_2 is equal to__________.

Ans. (6)

Sol.



$|z-3|=Re(z)$

let Z=x=iy

$\Rightarrow (x-3)^2+y^2=x^2$

$\Rightarrow x^2+9-6x+y^2=x^2$

$\Rightarrow y^2=6x-9$

$\Rightarrow y^2 = 6\left(x-\frac{3}{2}\right)$

$\Rightarrow$ z_1 and z_2 lie on the parabola mentioned in eq. (1)

arg $(z_1-z_2)=\frac{\pi}{4}$

$\Rightarrow$ Slope of PQ = 1

Let $P\left(\frac{3}{2}+\frac{3}{2}t_1^2, 3t_1\right)$ and $Q\left(\frac{3}{2}+\frac{3}{2}t_2^2, 3t_2\right)$

Slope of $PQ = \frac{3(t_2-t_1)}{\frac{3}{2}(t_2^2-t_1^2)} = 1$

$\Rightarrow \frac{2}{t_1+t_2}=1$

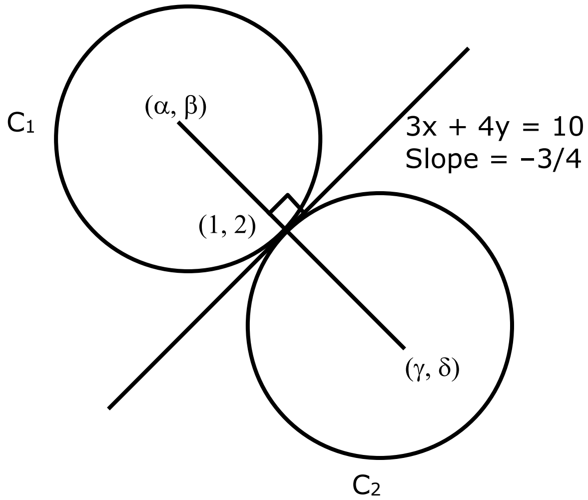
$\Rightarrow t_2+t_1=2$

Im $(z_1+z_2)=3t_1+3t_2=3(t_1+t_2)=3(2) = 6$

8. Two circles each of radius 5 units touch each other at the point $(1,2)$. If the equation of their common tangent is $4x + 3y = 10$, and $C_1(\alpha,\beta)$ and $C_2(\gamma,\delta), C_1 \neq C_2$ are their centres, then $|(\alpha+\beta)(\gamma+\delta)|$ is equal to. ______________.

Ans. **(40)**

Sol. Slope of line joining centres of circles $=\frac{4}{3}=\tan\theta$



$\Rightarrow \cos\theta = \frac{3}{5}, \sin\theta = \frac{4}{5}$

Now using parametric form

$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \pm 5$

$\oplus (x,y) = (1+5\cos\theta, 2+5\sin\theta)$

$(\alpha,\beta) = (4,6)$

$\Theta (x,y) = (\gamma,\delta) = (1-5\cos\theta, 2-5\sin\theta)$

$(\gamma,\delta) = (-2,-2)$

$\Rightarrow |(\alpha+\beta)(\gamma+\delta)| = |10\times(-4)| = 40$

9. If $\int \frac{2e^x+3e^{-x}}{4e^x+7e^{-x}} dx = \frac{1}{14}\left(ux + v\log_e\left(4e^x+7e^{-x}\right)\right) + C$, where C is a constant of integration, then u + v is equal to______________.

Ans. **(7)**

Sol. $\int \frac{2e^x}{4e^x+7e^{-x}} dx + 3\int \frac{e^{-x}}{4e^x+7e^{-x}} dx$

$= \int \frac{2e^{2x}}{4e^{2x}+7} dx + 3\int \frac{e^{-2x}}{4+7e^{-2x}} dx$

Let $4e^{2x} + 7 = T$| Let $4+7e^{-2x} = t$

$8e^{2x}dx = dT$ $-14e^{-2x}dx = dt$

$2e^{2x}dx = \frac{dT}{4}$ $e^{-2x}dx = -\frac{dt}{14}$

$\int \frac{dT}{4T} - \frac{3}{14}\int \frac{dt}{t}$

$= \frac{1}{4}\log T - \frac{3}{14}\log t + C$

$= \frac{1}{4}\log\left(4e^{2x}+7\right) - \frac{3}{14}\log\left(4+7e^{-2x}\right) + C$

$= \frac{1}{14}\left[\frac{1}{2}\log\left(4e^{x}+7e^{-x}\right) + \frac{13}{2}x\right] + C$

$u = \frac{13}{2}, v = \frac{1}{2} \Rightarrow u + v = 7$

$u=\frac{13}{2};v=\frac{1}{2}$

$\Rightarrow u+v=7$

10. $3 \times 7^{22} + 2 \times 10^{22} - 44$ when divided by 18 leaves the remainder____________.

Ans. **(15)**

Sol. $3(1+6)^{22} + 2\cdot(1+9)^{22} - 44 = (3+2-44) = 18.\,I$

$= -39 + 18.I$

$= (54-39) + 18(I-3)$

$= 15 + 18I$

$\Rightarrow$ Remainder = 15.