JEE MAIN
4^{th} Attempt

Mathematics

26^{th} August 2021 [SHIFT – 1]

QUESTION WITH SOLUTION

JEE | NEET | Foundation

SECTION - A

1. If the sum of an infinite GP a, ar, ar^2, ar^3, ... is 15 and the sum of the squares of its each term is 150, then the sum of ar^2, ar^4, ar^6, ... is :

(1) $\frac{1}{2}$

(2) $\frac{5}{2}$

(3) $\frac{25}{2}$

(4) $\frac{9}{2}$

Ans. (1)

Sol. Sum of infinite terms :

$\frac{a}{1-r} = 15$ .... (i)

Series formed by square of terms :

a^2, a^2r^2, a^2r^4, a^2r^6 ....

Sum $= \frac{a^2}{1-r^2} = 150$

$\Rightarrow \frac{a}{1-r}.\frac{a}{1+r} = 150 \qquad \Rightarrow \qquad 15 . \frac{a}{1+r} = 150$

$\Rightarrow \frac{a}{1+r} = 10$ .... (ii)

by (i) and (ii) a = 12; $r = \frac{1}{5}$

Now series : ar^2, ar^4, ar^6

Sum $= \frac{ar^2}{1-r^2} = \frac{12.\left(1/25\right)}{1-1/25} = \frac{1}{2}$

2. Let ABC be a triangle with A(–3, 1) and $\angle ACB = \theta$, $0 < \theta < \frac{\pi}{2}$. If the equation of the median through B is $2x + y - 3 = 0$ and the equation of angle bisector of C is $7x - 4y - 1 = 0$, then $\tan\theta$ is equal to :

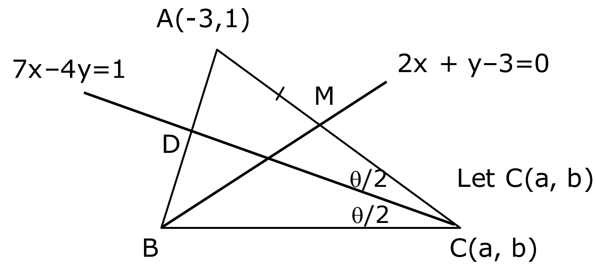
(1) $\frac{4}{3}$

(2) $\frac{1}{2}$

(3) 2

(4) $\frac{3}{4}$

Ans. (1)



Sol.

$\therefore$ M$\left(\frac{a-3}{2}, \frac{b+1}{2}\right)$ lies on 2x + y − 3 = 0

$\Rightarrow$ 2a + b = 11 .... (i)

$\because$ C lies on 7x − 4y = 1

$\Rightarrow$ 7a − 4b = 1 .... (ii)

$\therefore$ by (i) and (ii) : a = 3, b = 5

$\Rightarrow$ C (3, 5)

$\therefore$ m_{AC} = 2/3

Also, m_{CD} = 7/4

$$\Rightarrow \quad \tan\frac{\theta}{2} = \left|\frac{\frac{2}{3}-\frac{4}{4}}{1+\frac{14}{12}}\right| \quad \Rightarrow \quad \tan\frac{\theta}{2} = \frac{1}{2}$$

$$\Rightarrow \quad \tan\theta = \frac{2.\frac{1}{2}}{1-\frac{1}{4}} = \frac{4}{3}$$

3. Let $\theta \in \left(0, \frac{\pi}{2}\right)$. If the system of linear equations,

$(1 + \cos^2\theta)\,x + \sin^2\theta\,y + 4\sin3\theta\,z = 0$

$\cos^2\theta\,x + (1 + \sin^2\theta)\,y + 4\sin3\theta\,z = 0$

$\cos^2\theta\,x + \sin^2\theta\,y + (1 + 4\sin3\theta)\,z = 0$

has a non-trivial solution, then the value of θ is :

(1) $\frac{4\pi}{9}$

(2) $\frac{\pi}{18}$

(3) $\frac{5\pi}{18}$

(4) $\frac{7\pi}{18}$

Ans. (4)

Sol. Case − I

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 3\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 3\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 3\theta \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} 2 & \sin^2\theta & 4\sin 3\theta \\ 2 & 1+\sin^2\theta & 4\sin 3\theta \\ 1 & \sin^2\theta & 1+4\sin 3\theta \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & \sin^2\theta & 1+4\sin^3\theta \end{vmatrix} = 0$$

or $4\sin 3\theta = -2$

$\sin 3\theta = -\dfrac{1}{2}$

$\theta = \dfrac{7\pi}{18}$

4. The sum of solutions of the equation $\dfrac{\cos x}{1+\sin x} = |\tan 2x|$, $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) - \left\{\dfrac{\pi}{4}, -\dfrac{\pi}{4}\right\}$ is :

(1) $-\dfrac{11\pi}{30}$

(2) $-\dfrac{7\pi}{30}$

(3) $-\dfrac{\pi}{15}$

(4) $\dfrac{\pi}{10}$

Ans. **(1)**

Sol. $\dfrac{\cos x}{1+\sin x} = |\tan 2x|$

$\Rightarrow \dfrac{\cos^2 x/2 - \sin^2 x/2}{(\cos x/2 + \sin x/2)} = |\tan 2x|$

$\Rightarrow \tan^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) = \tan^2 2x$

$\Rightarrow 2x = n\pi \pm \left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)$

$\Rightarrow x = \dfrac{-3\pi}{10}, \dfrac{-\pi}{6}, \dfrac{\pi}{10}$

or sum $= \dfrac{-11\pi}{6}$.

5. Let A and B be independent events such that P(A) = p, P(B) = 2p. The largest value of p, for which P (exactly one of A, B occurs) = $\frac{5}{9}$, is :

(1) $\frac{4}{9}$

(2) $\frac{2}{9}$

(3) $\frac{1}{3}$

(4) $\frac{5}{12}$

Ans. **(4)**

Sol. P (Exactly one of A or B)

$= P\left(A \cap \bar{B}\right) + P\left(\bar{A} \cap B\right) = \frac{5}{9}$

$= P(A)\ P\left(\bar{B}\right) + P\left(\bar{A}\right)\ P(B) = \frac{5}{9}$

$\Rightarrow P(A)\ (1 - P(B)) + (1 - P(A))\ P(B) = \frac{5}{9}$

$\Rightarrow P\ (1 - 2p) + (1 - p)\ 2p = \frac{5}{9}$

$\Rightarrow 36p^2 - 27\,p + 5 = 0$

$\Rightarrow p = \frac{1}{3}$ or $\frac{5}{12}$

$P_{max} = \frac{5}{12}$

6. If the truth value of the Boolean expression $((p \vee q) \wedge (q \rightarrow r) \wedge (\sim r)) \rightarrow (p \wedge q)$ is false, then the truth values of the statements p, q, r respectively can be :

(1) FFT

(2) FTF

(3) TFT

(4) TFF

Ans. **(4)**

Sol.

p	**q**	**r**	$\underbrace{P \vee q}_{a}$	$\underbrace{q \rightarrow r}_{b}$	$a \wedge b$	**~r**	$\underbrace{a \wedge b \wedge (\sim r)}_{c}$	$\underbrace{p \wedge q}_{d}$	$c \rightarrow d$
T	F	T	T	T	T	F	F	F	T
F	F	T	F	T	F	F	F	F	T
T	F	F	T	T	T	T	T	F	F
F	T	F	T	F	F	T	F	F	T

7. The value of $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=0}^{2n-1}\frac{n^2}{n^2+4r^2}$ is :

(1) $\frac{1}{4}\tan^{-1}(4)$

(2) $\tan^{-1}(4)$

(3) $\frac{1}{2}\tan^{-1}(2)$

(4) $\frac{1}{2}\tan^{-1}(4)$

Ans. (4)

Sol. $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=0}^{2n-1}\frac{1}{1+4\left(\frac{r}{n}\right)^2}$

$\Rightarrow L = \int_0^2 \frac{1}{1+4x^2}dx$

$\Rightarrow L = \frac{1}{2}\tan^{-1}(2x)\Big|_0^2 \Rightarrow L = \frac{1}{2}\tan^{-1}4.$

8. Let y = y(x) be a solution curve of the differential equation $(y+1)\tan^2x\,dx + \tan x\,dy + ydx = 0$, $x \in \left(0,\frac{\pi}{2}\right)$. If $\lim\limits_{x\to0+} xy(x) = 1$, then the value of $y\left(\frac{\pi}{4}\right)$ is :

(1) $\frac{\pi}{4} - 1$

(2) $\frac{\pi}{4} + 1$

(3) $\frac{\pi}{4}$

(4) $-\frac{\pi}{4}$

Ans. (3)

Sol. $(y + 1)\tan^2x\,dx + \tan x\,dy + y\,dx = 0$

or $\frac{dy}{dx}+\frac{\sec^2 x}{\tan x}.y = -\tan x$

IF $= e^{\int\frac{\sec^2 x}{\tan x}dx} = e^{\ln\tan x} = \tan x$

$\therefore$ $y\tan x = -\int\tan^2 x dx$

or $y\tan x = -\tan x + x + C$

or $y = -1 + \frac{x}{\tan x}+\frac{C}{\tan x}$

or $\lim\limits_{x\to0} xy = -x + \frac{x^2}{\tan x}+\frac{Cx}{\tan x} = 1$

or C = 1

$y(x) = \cot x + x\cot x - 1$

$y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$

9. The mean and standard deviation of 20 observations were calculated as 10 and 2.5 respectively. It was found that by mistake one data value was taken as 25 instead of 35. If α and $\sqrt{\beta}$ are the mean and standard deivation respectively for correct data, then (α, β) is :

(1) (11, 25)
(2) (11, 26)
(3) (10.5, 26)
(4) (10.5, 25)

Ans. **(3)**

Sol. Given :

Mean $(\bar{x}) = \frac{\sum x_i}{20} = 10$

or $\sum x_i = 200$ (incorrect)

or $200 - 25 + 35 = 210 = \sum x_i$ (correct)

Now correct $\bar{x} = \frac{210}{20} = 10.5$

again given S.D. = 2.5 (σ)

$\sigma^2 = \frac{\sum x_i^2}{20} - (10)^2 = (2.5)^2$

or $\sum x_i^2 = 2125$ (incorrect)

or $\sum x_i^2 = 2125 - 25^2 + 35^2$

$= 2725$ (Correct)

$\therefore$ correct $\sigma^2 = \frac{2725}{20} - (10.5)^2$

$\sigma^2 = 26$

or $\sigma = \sqrt{26}$

$\therefore \alpha = 10.5, \beta = 26$

10. Let $f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$, $0 < x < 1$. Then :

(1) $(1 + x)^2 f'(x) + 2(f(x))^2 = 0$
(2) $(1 - x)^2 f'(x) + 2(f(x))^2 = 0$
(3) $(1 + x)^2 f'(x) - 2(f(x))^2 = 0$
(4) $(1 - x)^2 f'(x) - 2(f(x))^2 = 0$

Ans. **(2)**

Sol. $f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$

$\cot^{-1}\sqrt{\frac{1-x}{x}} = \sin^{-1}\sqrt{x}$

or $f(x) = \cos(2\tan^{-1}\sqrt{x})$

$= \cos\tan^{-1}\left(\frac{2\sqrt{x}}{1-x}\right)$

$f(x) = \frac{1-x}{1+x}$

Now $f'(x) = \frac{-2}{(1+x)^2}$

or $f'(x)\,(1-x)^2 = -2\left(\frac{1-x}{1+x}\right)^2$

or $(1-x)^2 f'(x) + 2(f(x))^2 = 0.$

11. The sum of the series $\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{2^2}{x^4+1}+.....+\frac{2^{100}}{x^{2^{100}}+1}$ when $x = 2$ is :

(1) $1 - \frac{2^{101}}{4^{101}-1}$

(2) $1 + \frac{2^{101}}{4^{101}-1}$

(3) $1 - \frac{2^{100}}{4^{100}-1}$

(4) $1 + \frac{2^{101}}{4^{101}-1}$

Ans. **(1)**

Sol. $S = \frac{1}{x+1}+\frac{2}{x^2+1}+\frac{2^2}{x^4+1}+.....+\frac{2^{100}}{x^{2^{100}}+1}$

$S + \frac{1}{1-x} = \frac{1}{1-x}+\frac{1}{x+1}+.... = \frac{2}{1-x^2}+\frac{2}{1+x^2}+....$

$S + \frac{1}{1-x} = \frac{2^{101}}{1-x^{2^{101}}}$

Put $x = 2$

$S = 1 - \frac{2^{101}}{2^{2^{101}}-1}$

Not in option (BONUS)

12. If ${}^{20}C_r$ is the co-efficient of x^r in the expansion of $(1 + x)^{20}$, then the value of $\sum_{r=0}^{20} r^2\ {}^{20}C_r$ is equal to :

(1) 420×2^{19}
(2) 420×2^{18}
(3) 380×2^{18}
(4) 380×2^{19}

Ans. **(2)**

Sol. $\sum_{r=0}^{20} r^2.\ {}^{20}C_r$

$\sum \left(4(r-1)+r\right).\ {}^{20}C_r$

$\sum r(r-1).\frac{20\times19}{r(r-1)}.{}^{18}C_r + r.\frac{20}{r}\sum{}^{19}C_{r-1}$

$\Rightarrow$ $20 \times 19.2^{18} + 20.2^{19}$

$\Rightarrow$ 420×2^{18}

13. A plane P contains the line $x + 2y + 3z + 1 = 0 = x - y - z - 6$, and is perpendicular to the plane $-2x + y + z + 8 = 0$. Then which of the following points lies on P?

(1) (1, 0, 1)

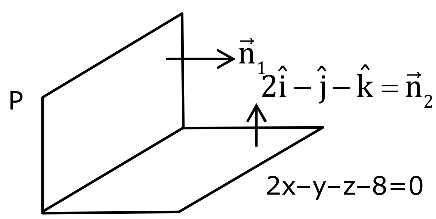
(2) (2, –1, 1)

(3) (–1, 1, 2)

(4) (0, 1, 1)

Ans. **(4)**

Sol. Equation of plane P can be assumed as



$P : x + 2y + 3z + 1 + \lambda(x - y - z - 6) = 0$

$\Rightarrow P : (1 + \lambda)x + (2 - \lambda)y + (3 - \lambda)z + 1 - 6\lambda = 0$

$\Rightarrow \vec{n}_1 = (1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (3-\lambda)\hat{k}$

$\therefore \vec{n}_1.\vec{n}_2 = 0$

$\Rightarrow \quad 2(1 + \lambda) - (2 - \lambda) - (3 - \lambda) = 0$

$\Rightarrow \quad 2 + 2\lambda - 2 + \lambda - 3 + \lambda = 0 \Rightarrow \lambda = \frac{3}{4}$

$\Rightarrow \quad P : \frac{7x}{4} + \frac{5}{4}y + \frac{9z}{4} - \frac{14}{4} = 0$

$\Rightarrow \quad 7x + 5y + 9z = 14$

(0, 1, 1) lies on P

14. On the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ let P be a point in the second quadrant such that the tangent at P to the ellipse is perpendicular to the line $x + 2y = 0$. Let S and S' be the foci of the ellipse and e be its eccentricity. If A is the area of the triangle SPS' then, the value of $(5 - e^2)$. A is :

(1) 24

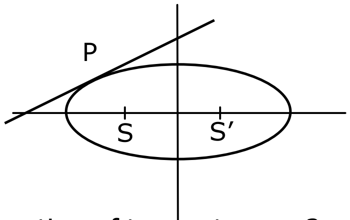
(2) 6

(3) 14

(4) 12

Ans. **(2)**

Sol.



Equation of tangent : y = 2x + 6
at P
$\therefore$ P (−8/3, 2/3)

$e = \frac{1}{\sqrt{2}}$

S & S' = (−2, 0) & (2, 0)

Area of ΔSPS' = $\frac{1}{2} \times 4 \times \frac{2}{3}$

A = $\frac{4}{3}$

$\therefore$ $(5 - e^2)A = \left(5 - \frac{1}{2}\right)\frac{4}{3} = 6$

15. The value of $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right)^{\frac{1}{2}} dx$ is :

(1) $\log_e 4$
(2) $\log_e 16$
(3) $4 \log_e\left(3 + 2\sqrt{2}\right)$
(4) $2 \log_e 16$

Ans. (2)

Sol. $I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right)^2\right)^{\frac{1}{2}} dx$

$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left|\frac{4x}{x^2-1}\right| dx \quad \Rightarrow I = 2.4 \int_{0}^{1/\sqrt{2}} \left|\frac{x}{x^2-1}\right| dx$

$\Rightarrow I = -4 \int_{0}^{1/\sqrt{2}} \frac{2x}{x^2-1} dx \quad \Rightarrow I = -4 \left|x^2 - 1\right|_{0}^{1/\sqrt{2}}$

$\Rightarrow$ I = 4 ln 2 $\quad \Rightarrow$ I = ln 16

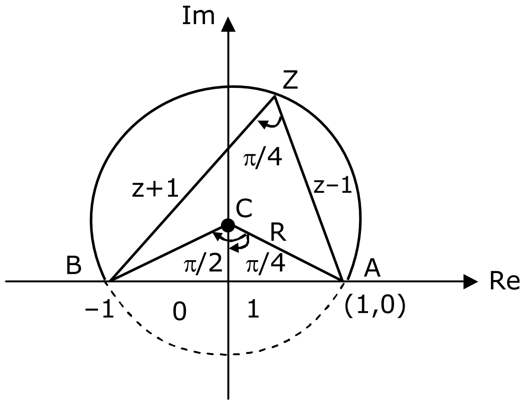
16. The equation $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ represents a circle with :

(1) centre at (0, −1) and radius $\sqrt{2}$

(2) centre at (0, 1) and radius 2
(3) centre at (0, 1) and radius $\sqrt{2}$
(4) centre at (0, 0) and radius $\sqrt{2}$

Ans. (3)

Sol.



In ΔOAC

$\sin\left(\frac{\pi}{4}\right) = \frac{1}{AC}$

$\Rightarrow AC = \sqrt{2}$

Also, $\tan\frac{\pi}{4} = \frac{OA}{OC} = \frac{1}{OC}$

$\Rightarrow OC = 1$

$\therefore$ centre (0, 1) ; Radius = $\sqrt{2}$

17. If a line along a chord of the circle $4x^2 + 4y^2 + 120x + 675 = 0$, passes through the point (–30, 0) and is tangent to the parabola $y^2 = 30x$, then the length of this chord is :

(1) 5
(2) $3\sqrt{5}$
(3) 7
(4) $5\sqrt{3}$

Ans. (2)

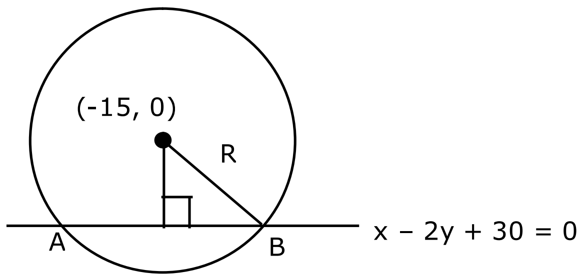
Sol. Equation of tangent to $y^2 = 30x$

$y = mx + \frac{30}{4m}$

Pass thru (–30, 0) : $a = -30\,m + \frac{30}{4m} \Rightarrow m^2 = ¼$

$\Rightarrow m = \frac{1}{2}$ or $m = -\frac{1}{2}$

At $m = \frac{1}{2}$: $y = \frac{x}{2} + 15 \quad \Rightarrow x - 2y + 30 = 0$



$P = \frac{15}{\sqrt{5}}$

$\ell_{AB} = 2\sqrt{R^2 - P^2} \qquad = 2\sqrt{\frac{225}{4} - \frac{225}{5}}$

$\Rightarrow \ell_{AB} = 30\sqrt{\frac{1}{20}} \qquad = \frac{15}{\sqrt{5}} = 3\sqrt{5}$

18. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$. If $\vec{c}$ is a vector such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a}\,.\,\vec{c} = 3$, then $\vec{a}.\left(\vec{b} \times \vec{c}\right)$ is equal to :

(1) −2
(2) 6
(3) 2
(4) −6

Ans. **(1)**

Sol. $|\vec{a}| = \sqrt{3}$; $\vec{a}.\vec{c} = 3$; $\vec{a} \times \vec{b} = -2\hat{i} + \hat{j} + \hat{k}$, $\vec{a} \times \vec{c} = \vec{b}$

Cross with $\vec{a}$.

$\vec{a} \times \left(\vec{a} \times \vec{c}\right) = \vec{a} \times \vec{b}$

$\Rightarrow (\vec{a}.\vec{c})\vec{a} - a^2\,\vec{c} = \vec{a} \times \vec{b}$

$\Rightarrow 3\vec{a} - 3\vec{c} = -2\hat{i} + \hat{j} + \hat{k}$

$\Rightarrow 3\hat{i} + 3\hat{j} + 3\hat{k} - 3\vec{c} = -2\hat{i} + \hat{j} + \hat{k}$

$\Rightarrow \vec{c} = \frac{5\hat{i}}{3} + \frac{2\hat{j}}{3} + \frac{2\hat{k}}{3}$

$\therefore \vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c} = \frac{-10}{3} + \frac{2}{3} + \frac{2}{3} = -2$

19. If $A = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$, $i = \sqrt{-1}$ and $Q = A^T BA$, then the inverse of the matrix $A\,Q^{2021}\,A^T$ is equal to :

(1) $\begin{pmatrix} \frac{1}{\sqrt{5}} & -2021 \\ 2021 & \frac{1}{\sqrt{5}} \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 0 \\ 2021\,i & 1 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 0 \\ -2021\,i & 1 \end{pmatrix}$

(4) $\begin{pmatrix} 1 & -2021\,i \\ 0 & 1 \end{pmatrix}$

Ans. **(3)**

Sol. $AA^T = \begin{pmatrix} \frac{1}{5} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$

$AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$Q^2 = A^TBA\ A^TBA = A^TBIBA$

$\Rightarrow Q^2 = A^TB^2A$

$Q^3 = A^TB^2AA^TBA \qquad \Rightarrow Q^3 = A^TB^3A$

Similarly : $\quad Q^{2021} = A^TB^{2021}\,A$ .... (1)

Now $\quad B^2 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}$

$B^3 = \begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \Rightarrow B^3 = \begin{pmatrix} 1 & 0 \\ 3i & 1 \end{pmatrix}$

Similarly $B^{2021} = \begin{pmatrix} 1 & 0 \\ 2021i & 1 \end{pmatrix}$

$\therefore\ AQ^{2021}\,A^T = AA^TB^{2021}\,AA^T = IB^{2021}\,I$

$\Rightarrow AQ^{2021}\,A^T = B^{2021} = \begin{pmatrix} 1 & 0 \\ 2021i & 1 \end{pmatrix}$

$\therefore (AQ^{2021}\,A^T)^{-1} = \begin{pmatrix} 1 & 0 \\ 2021i & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -2021i & 1 \end{pmatrix}$

20. Out of all the patients in a hospital 89% are found to be suffering from heart ailment and 98% are suffering from lungs infection. If K% of them are suffering from both ailments, then K can not belong to the set :

(1) {80, 83, 86, 89}
(2) {79, 81, 83, 85}
(3) {84, 87, 90, 93}
(4) {84, 86, 88, 90}

Ans. **(2)**

Sol. $n(A \cup B) \ge n(A) + n(B) - n(A \cap B)$

$100 \ge 89 + 98 - n(A \cup B)$

$n(A \cup B) \ge 87$

$87 \le n(A \cup B) \le 89$

Option (2)

Section B

1. The sum of all integral values of k $(k \neq 0)$ for which the equation $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$ in x has no real roots, is _______.

Ans. (66)

Sol. $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$

$x \in R - \{1, 2\}$

$\Rightarrow k(2x - 4 - x + 1) = 2(x^2 - 3x + 2)$

$\Rightarrow k(x - 3) = 2(x^2 - 3x + 2)$

for $x \neq 3$, $\quad k = 2\left(x - 3 + \frac{2}{x-3} + 3\right)$

$x - 3 + \frac{2}{x-3} \geq 2\sqrt{2} \forall < -3$

$\& \ x - 3 + \frac{2}{x-3} \leq -2\sqrt{2}, \ \forall x < -3$

$\Rightarrow 2\left(x - 3 + \frac{2}{x-3} + 3\right) \in \left(-\infty, 6 - 4\sqrt{2}\right] \cup \left[6 + 4\sqrt{2}, \infty\right)$

for no real roots

$k \in (6 - 4\sqrt{2}, 6 + 4\sqrt{2}) - \{0\}$

Integral $k \in \{1, 2, \ldots\ldots 11\}$

Sum of k = 66

2. The locus of a point, which moves such that the sum of squares of its distances from the points (0, 0), (1, 0), (0, 1), (1, 1) is 18 units, is a circle of diameter d. Then d^2 is equal to _______.

Ans. (16)

Sol. Let P(x, y)

$x^2 + y^2 + x^2 + (y - 1)^2 + (x - 1)^2 + y^2 + (x - 1)^2 + (y - 1)^2$;

$\Rightarrow 4(x^2 + y^2) - 4y - 4x = 14$

$\Rightarrow x^2 + y^2 - x - y - \frac{7}{2} = 0$

$d = 2\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{7}{2}}$

$\Rightarrow d^2 = 16$

3. If y = y(x) is an implicit function of x such that $\log_e(x+y) = 4xy$, then $\frac{d^2y}{dx^2}$ at x = 0 is equal to _______.

Ans. (40)

Sol. $\ln(x + y) = 4xy \qquad$ (At x = 0, y = 1)

$x + y = e^{4xy}$

$\Rightarrow 1 + \frac{dy}{dx} = e^{4xy}\left(4x\frac{dy}{dx} + 4y\right)$

At $x = 0$ $\quad \frac{dy}{dx} = 3$

$$\frac{d^2y}{dx^2} = e^{4xy}\left(4x\frac{dy}{dx} + 4y\right)^2 + e^{4xy}\left(4x\frac{d^2y}{dx^2} + 4y\right)$$

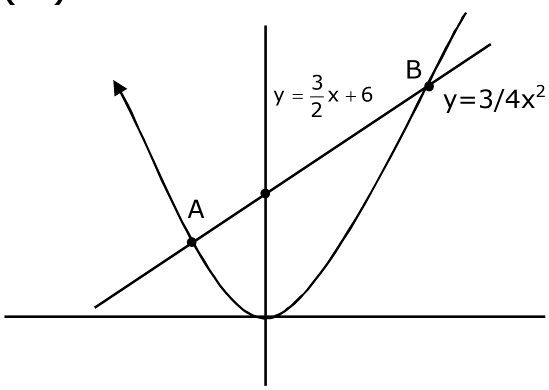
At $x = 0$, $\frac{d^2y}{dx^2} = e^0(4)^2 + e^0(24)$

$\Rightarrow \frac{d^2y}{dx^2} = 40$

4. The area of the region $S = \{(x, y) : 3x^2 \le 4y \le 6x + 24\}$ is ________.

Ans. **(27)**

Sol.



For A & B

$3x^2 = 6x + 24 \Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow x = -2, 4$

$$\text{Area} = \int_{-2}^{4}\left(\frac{3}{2}x + 6 - \frac{3}{4}x^2\right)dx$$

$$= \left[\frac{3x^2}{4} + 6x - \frac{x^3}{4}\right]_{-2}^{4} = 27$$

5. Let $a, b \in \mathbf{R}$, $b \ne 0$. Define a function

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x-1) & \text{for } \le 0 \\ \frac{\tan 2x - \sin 2x}{bx^3} & \text{for } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then $10 - ab$ is equal to __________.

Ans. **(14)**

Sol. $$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x-1) & \text{for } \le 0 \\ \frac{\tan 2x - \sin 2x}{bx^3} & \text{for } x > 0 \end{cases}$$

For continuity at '0'

$\lim_{x \to 0^+} f(x) = f(0)$

$\Rightarrow \lim_{x \to 0^+} \frac{\tan 2x - \sin 2x}{bx^3} = -a$

$\Rightarrow \lim_{x \to 0^+} \frac{\frac{8x^3}{3} + \frac{8x^3}{3!}}{bx^3} = -a$

$\Rightarrow 8\left(\frac{1}{3} + \frac{1}{3!}\right) = -ab$

$\Rightarrow 4 = -ab$

$\Rightarrow 10 - ab = 14$

6. If ${}^1P_1 + 2\,.{}^2P_2 + 3\,.{}^3P_3 + \ldots\ldots\; 15\,.{}^{15}P_{15} = {}^qP_r - s,\ 0 \le s \le 1$, then ${}^{q+s}C_{r-s}$ is equal to __________.

Ans. **(136)**

Sol. ${}^1P_1 + 2\,.{}^2P_2 + 3\,.{}^3P_3 + \ldots\ldots\; 15\,.{}^{15}P_{15}$

$= 1! + 2.\ 2! + 3.\ 3! + \ldots\ldots\; 15 \times 15!$

$= \sum_{r=1}^{15} (r + 1 - 1) r!$

$= \sum_{r=1}^{15} (r+1)! - (r)!$

$= 16! - 1$

$= {}^{16}P_{16} - 1$

$\Rightarrow q = r = 16,\ s = 1$

${}^{q+s}C_{r-s} = {}^{17}C_{15} = 136$

7. A wire of length 36 m is cut into two pieces, one of the pieces is bent to form a square and the other is bent to form a circle. If the sum of the areas of the two figures is minimum, and the circumference of the circle is k (meter), then $\left(\frac{4}{\pi} + 1\right)k$ is equal to ______.

Ans. **(36)**

Sol. Let $x + y = 36$

x is perimeter of square and y is perimeter of circle side of square = $x/4$

radius of circle = $\frac{y}{2\pi}$

Sum Areas = $\left(\frac{x}{4}\right)^2 + \pi\left(\frac{y}{2\pi}\right)^2$

$= \frac{x^2}{16} + \frac{(36 - x)^2}{4\pi}$

For min Area :

$x = \frac{144}{\pi + 4}$

$\Rightarrow$ Radius = y = $36 - \dfrac{144}{\pi + 4}$

$\Rightarrow k = \dfrac{36\pi}{\pi + 4}$

$\left(\dfrac{4}{\pi} + 1\right)k = 36$

8. Let the line L be the projection of the line :

$\dfrac{x-1}{2} = \dfrac{y-3}{1} = \dfrac{z-4}{2}$

in the plane x – 2y – z = 3. If d is the distance of the point (0, 0, 6) from L, then d^2 is equal to ______.

Ans. **(26)**

Sol. $L_1 : \dfrac{x-1}{2} = \dfrac{y-3}{1} = \dfrac{z-4}{2}$

for foot of $\perp$ r of (1, 3, 4) on x – 2y – z – 3 = 0

(1 + t) – 2 (3 – 2t) – (4 – t) – 3 = 0

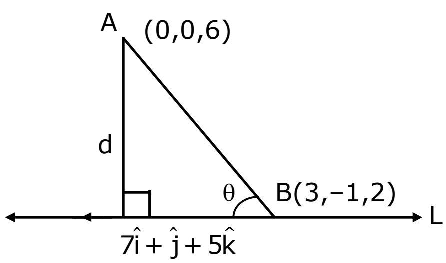
$\Rightarrow$ t = 2

So foot of $\perp$ r $\hat{=}$ (3, –1, 2)

& point of intersection of L_1 with plane

is (–11, –3, –8)

dr's of L is < 14, 2, 10> $\cong$ < 7, 1, 5>



$$d = AB\sin\theta = \left| \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -4 \\ 7 & 1 & 5 \end{vmatrix}}{\sqrt{7^2 + 1^2 + 5^2}} \right|$$

$$\Rightarrow d^2 = \frac{1^2 + (43)^2 + (10)^2}{49 + 1 + 25} = 26$$

9. Let $z = \dfrac{1 - i\sqrt{3}}{2}$, $i = \sqrt{-1}$. Then the value of

$21 + \left(z + \dfrac{1}{z}\right)^3 + \left(z^2 + \dfrac{1}{z^2}\right)^3 + \left(z^3 + \dfrac{1}{z^3}\right)^3 + ... + \left(z^{21} + \dfrac{1}{z^{21}}\right)^3$ is ______________.

Ans. **(13)**

Sol. $Z = \frac{1-\sqrt{3}i}{2} = e^{-i\frac{\pi}{3}}$

$$z^r + \frac{1}{z^r} = 2\cos\left(-\frac{\pi}{3}\right)r = 2\cos\frac{r\pi}{3}$$

$$\Rightarrow \ 21 + \sum_{r=1}^{21}\left(z^r + \frac{1}{z^r}\right)^3 = 8\left(\cos^3\frac{r\pi}{3}\right) = 2\left(\cos r\pi + 3\cos\frac{r\pi}{3}\right)$$

$$\Rightarrow \ 21 + \left(z + \frac{1}{2}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \ldots.\left(z^{21} + \frac{1}{z^{21}}\right)^3$$

$$= \ 21 + \sum_{r=1}^{21}\left(z^r + \frac{1}{z^r}\right)^3$$

$$= \ 21 + \sum_{r=1}^{21}\left(2\cos r\pi + 6\cos\frac{r\pi}{3}\right)$$

= 21 – 2 – 6

= 13

10. The number of three-digit even numbers, formed by the digits 0, 1, 3, 4, 6, 7 if the repetition of digits is not allowed, is ___________.

Ans. (52)

Sol. (i) When '0' is at unit place

$\uparrow$	$\uparrow$	0
5	4	

Number of numbers = 20

(ii) When 4 or 6 are at unit place

OX		4,6
4×4		$\uparrow$ 2

Number of numbers = 32

So number of numbers = 52