JEE MAIN
4th Attempt

MATHEMATICS
1st September 2021 [SHIFT – 2]

QUESTION WITH SOLUTION

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SECTION - A

1. Let P_1, P_2 ..., P_{15} be 15 points on a circle. The number of distinct triangles formed by points P_i, P_j, P_k such that $i+j+k \neq 15$, is:

(1) 12 (2) 419 (3) 455 (4) 443

Ans. (4)

Sol. Total number of triangles $= {}^{15}C_3$

i + j + k = 15 (Given)

5 Cases

i	j	k
1	2	12
1	3	11
1	4	10
1	5	9
1	6	8

4 Cases

i	j	k
2	3	10
2	4	9
2	5	8
2	6	7

3 case

i	j	k
3	4	8
3	5	7

1 case

i	j	k
4	5	6

Number of possible triangle using the vertices P_i, P_j, P_k such that $i + j + k \neq 15$ is equal to

${}^{15}C_3 - 12 = 443$

2. The function $f(x)$, that satisfies the condition $f(x) = x + \int_0^{\pi/2} \sin x \cdot \cos y \, f(y)\, dy$, is:

(1) $x + (\pi - 2)\sin x$

(2) $x + \frac{\pi}{2}\sin x$

(3) $x + \frac{2}{3}(\pi - 2)\sin x$

(4) $x + (\pi + 2)\sin x$

Ans. (1)

Sol. $f(x) = x + \int_0^{\pi/2} \sin x \cos y f(y)\, dy$

$f(x) = x + \sin x \underbrace{\int_0^{\pi/2} \cos y f(y)\, dy}_{K}$

$\Rightarrow f(x) = x + K \sin x$

$\Rightarrow f(y) = y + K \sin y$

Now $K = \int_0^{\pi/2} \cos y (y + K \sin y)\, dy$

$K = \underbrace{\int_0^{\pi/2} y \cos dy}_{\text{Apply IBP}} + \underbrace{\int_0^{\pi/2} \cos y \sin y dy}_{\text{Put } \sin y = t}$

$K = (y \sin y)_0^{\pi/2} - \int_0^{\pi/2} \sin y \, dy + K\int_0^1 t \, dt$

$\Rightarrow K = \frac{\pi}{2} - 1 + K\left(\frac{1}{2}\right)$

$\Rightarrow k = \pi - 2$

So $f(x) = x + (\pi - 2)\sin x$

3. If y=y(x) is the solution curve of the differential equation

$x^2dy + \left(y - \frac{1}{x}\right)dx = 0;\ x > 0$ and $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to :

(1) $3 + \frac{1}{\sqrt{e}}$ (2) $\frac{3}{2} - \frac{1}{\sqrt{e}}$ (3) $3 + e$ (4) $3 - e$

Ans. (4)

Sol. $x^2dy + \left(y - \frac{1}{x}\right)dx = 0 : x > 0, y(1) = 1$

$$x^2dy + \frac{(xy-1)}{x}dx = 0$$

$$x^2dy = \frac{(xy-1)}{x}dx$$

$$\frac{dy}{dx} = \frac{1-xy}{x^3}$$

$$\frac{dy}{dx} = \frac{1}{x^3} - \frac{y}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2}.y = \frac{1}{x^3}$$

If $e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$

$$ye^{-\frac{1}{x}} = \int \frac{1}{x^3}.e^{-\frac{1}{x}}$$

$$ye^{-\frac{1}{x}} = e^{-\frac{1}{x}}\left(1 + \frac{1}{x}\right) + C$$

$$1.e^{-1} = e^{-1}(2) + c$$

$$C = -e^{-1} = -\frac{1}{e}$$

$$ye^{-\frac{1}{x}} = e^{-\frac{1}{x}}\left(1 + \frac{1}{x}\right) - \frac{1}{e}$$

$$y\left(\frac{1}{2}\right) = 3 - \frac{1}{e} \times e^2$$

$$y\left(\frac{1}{2}\right) = 3 - e$$

4. The distance of line 3y–2z–1=0=3x–z+4 from the point (2, –1,6) is:

(1) $2\sqrt{6}$ (2) $4\sqrt{2}$ (3) $2\sqrt{5}$ (4) $\sqrt{26}$

Ans. (1)

Sol. $3y - 2z - 1 = 0 = 3x - z + 4$

$3y - 2z - 1 = 0$ D.R's $\Rightarrow$ (0, 3, –2)

$3x - z + 4 = 0$ D.R's $\Rightarrow$ (3, –1, 0)

Let DR's of given line are a, b, c

Now $3b - 2c = 0$ & $3a - c = 0$

$\therefore$ $6a = 3b = 2c$

a : b : c = 3 : 6 : 9

Any point on line

$3K - 1, 6K + 1, 9K + 1$

Now $3(3K - 1) + 6(6K + 1) + 9(9K + 1) = 0$

$\Rightarrow K = \frac{1}{3}$

Point on line $\Rightarrow$ (0, 3, 4)

Given point (2, –1, 6)

$\Rightarrow$ Distance $= \sqrt{4+16+4} = 2\sqrt{6}$

5. The number of pairs (a, b) of real numbers, such that whenever α is a root of the equation $x^2 + ax + b = 0, \alpha^2 - 2$ is also a root of this equation is:

(1) 6 (2) 4 (3) 8 (4) 2

Ans. (1)

Sol. Consider the equation $x^2 + ax + b = 0$

If has two roots (not necessarily real α & β)

Either $\alpha = \beta$ or $\alpha \neq \beta$

Case (1) If $\alpha = \beta$, then it is repeated root. Given

that $\alpha^2 - 2$ is also a root

So, $\alpha = \alpha^2 - 2 \Rightarrow (\alpha + 1)(\alpha - 2) = 0$

$\Rightarrow \alpha = -1$ or $\alpha = 2$

When $\alpha = -1$ then $(a, b) = (2, 1)$

$\alpha = 2$ then $(a, b) = (-4, 4)$

Case (2) If $\alpha \neq \beta$ Then

(I) $\alpha = \alpha^2 - 2$ and $\beta = \beta^2 - 2$

Here $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$

Hence $(\alpha, \beta) = (-(\alpha + \beta), \alpha\beta)$

$= (-1, -2)$

(II) $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$

Then $\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$

Since $\alpha \neq \beta$ we get $\alpha + \beta = \beta^2 + \alpha^2 - 4$

$\alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta - 4$

Thus $-1 = 1 - 2\alpha\beta - 4$ which implies

$\alpha\beta = -1$ Therefore $(a,b) = (-(\alpha + \beta), \alpha\beta)$

$= (1, -1)$

(III) $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$

$\Rightarrow \alpha = -\beta$

Thus $\alpha = 2, \beta = -2$

$\alpha = -1, \beta = 1$

Therefore $(a,b) = (0, -4)$ & $(0, -1)$

(IV) $\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$ is same as (III)

Therefore we get 6 pairs of (a, b)

Which are $(2, 1), (-4, 4), (-1, -2), (1, -1)\ (0, -4)$

6. Let $S_n = 1\cdot(n-1) + 2\cdot(n-2) + 3\cdot(n-3) + \ldots + (n-1)\cdot 1, n \geq 4$.

The sum $\sum_{n=4}^{\infty}\left(\frac{2S_n}{n!} - \frac{1}{(n-2)!}\right)$ is equal to:

(1) $\frac{e-1}{3}$ (2) $\frac{e-2}{6}$ (3) $\frac{e}{6}$ (4) $\frac{e}{3}$

Ans. (1)

Sol. Let $T_r = r(n - r)$

$T_r = nr - r^2$

$$\Rightarrow S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left(nr - r^2\right)$$

$$S_n = \frac{n.(n)(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow S_n = \frac{n(n-1)(n+1)}{6}$$

Now $\displaystyle\sum_{r=4}^{\infty}\left(\frac{2S_n}{n!} - \frac{1}{(n-2)!}\right)$

$$= \sum_{r=4}^{\infty}\left(2.\frac{n(n-1)(n+1)}{6.n(n-1)(n-2)!} - \frac{1}{(n-2)!}\right)$$

$$= \sum_{r=4}^{\infty}\left(\frac{1}{3}\left(\frac{n-2+3}{(n-2)!}\right) - \frac{1}{(n-2)!}\right)$$

$$= \sum_{r=4}^{\infty}\frac{1}{3}.\frac{1}{(n-3)!} = \frac{1}{3}(e-1)$$

7. $\cos^{-1}(\cos(-5))+\sin^{-1}(\sin(6))-\tan^{-1}(\tan(12))$ is equal to:

(The inverse trigonometric functions take the principal values)

(1) $3\pi-11$ (2) $4\pi-11$ (3) $3\pi+1$ (4) $4\pi-9$

Ans. (2)

Sol. $\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6)) - \tan^{-1}(\tan(12))$

$\Rightarrow (2\pi - 5) + (6 - 2\pi) - (12 - 4\pi)$

$\Rightarrow 4\pi - 11.$

8. Let the acute angle bisector of the two planes $x-2y-2z+1=0$ and $2x-3y-6z+1=0$ be the plane P. Then which of the following points lies on P?

(1) $(4, 0, -2)$ (2) $\left(-2, 0, -\frac{1}{2}\right)$ (3) $\left(3, 1, -\frac{1}{2}\right)$ (4) $(0, 2, -4)$

Ans. (2)

Sol. $P_1 : x - 2y - 2z + 1 = 0$

$P_2 : 2x - 3y - 6z + 1 = 0$

$$\left|\frac{x-2y-2z+1}{\sqrt{1+4+4}}\right| = \left|\frac{2x-3y-6z+1}{\sqrt{2^2+3^2+6^2}}\right|$$

$$\frac{x-2y-2z+1}{3}=\pm\frac{2x-3y-6z+1}{7}$$

Since $a_1a_2 + b_1b_2 + c_1c_2 = 20 > 0$

$\therefore$ Negative sign will give

acute bisector

$7x - 14y - 14z + 7 = -[6x - 9y - 18z + 3]$

$\Rightarrow 13x - 23y - 32z + 10 = 0$

$\left(-2, 0, -\frac{1}{2}\right)$ satisfy it $\therefore$

9. The area, enclosed by curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ and the lines x=0, $x = \frac{\pi}{2}$, is:

(1) $2\sqrt{2}\left(\sqrt{2}+1\right)$ (2) $4\left(\sqrt{2}-1\right)$ (3) $2\left(\sqrt{2}+1\right)$ (4) $2\sqrt{2}\left(\sqrt{2}-1\right)$

Ans. (4)

Sol. $A = \int_0^{\pi/2}\left((\sin x + \cos x) - |\cos x - \sin x|\right)dx$

$A = \int_0^{\pi/4}\left((\sin x + \cos x) - (\cos x - \sin x)\right)dx$

$+\int_{\pi/4}^{\pi/2}\left((\sin x + \cos x) - (\sin x - \cos x)\right)dx$

$A = 2\int_0^{\pi/4}\sin x\,dx + 2\int_{\pi/4}^{\pi/2}\cos x\,dx$

$A = -2\left(\frac{1}{\sqrt{2}} - 1\right) + 2\left(1 - \frac{1}{\sqrt{2}}\right)$

$A = 4 - 2\sqrt{2} = 2\sqrt{2}\left(\sqrt{2}-1\right)$

10. Which of the following is equivalent to the Boolean expression $p \wedge \sim q$?

(1) $\sim(p \to \sim q)$ (2) $\sim p \to \sim q$ (3) $\sim(q \to p)$ (4) $\sim(p \to q)$

Ans. (4)

Sol.

p	q	~p	~q	p–q	~(p→q)	q→p	~(q→p)
T	T	F	F	T	F	T	F
T	F	F	T	F	T	T	F

F	T	T	F	T	F	F	T
F	F	T	T	T	F	T	F

$p \wedge \sim q$	$\sim p \to \sim q$	$p \to \sim q$	$\sim(p \to \sim q)$
F	T	F	T
T	T	T	F
F	F	T	F
F	T	T	F

$p \wedge \sim q \equiv \sim (p \to q)$

11. If n is the number of solutions of the equation $2\cos x\left(4\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)-1\right)=1, x \in [0, \pi]$ and S is the sum of all these solutions, than the ordered pair (n, S) is:

(1) $(3, 13\pi/9)$ (2) $(2, 8\pi/9)$ (3) $(3, 5\pi/3)$ (4) $(2, 2\pi/3)$

Ans. (1)

Sol. $$2\cos x\left(4\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)-1\right)=1$$

$$2\cos x\left(4\left(\sin^2\frac{\pi}{4}-\sin^2 x\right)-1\right)=1$$

$$2\cos x\left(4\left(\frac{1}{2}-\sin^2 x\right)-1\right)=1$$

$$2\cos x\left(2-4\sin^2 x-1\right)=1$$

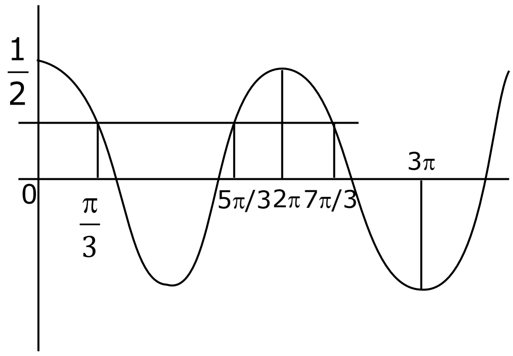
$$2\cos x\left(1-4\sin^2 x\right)=1$$

$$2\cos x\left(4\cos^2 x-3\right)=1$$

$$4\cos^3 x-3\cos x=\frac{1}{2}$$

$$\cos 3x=\frac{1}{2}$$

$$x \in [0,\pi] \therefore 3x \in [0,3\pi]$$



12. The function $f(x) = x^3 - 6x^2 + ax + b$ is such that $f(2) = f(4) = 0$. Consider two statements.

(S1) there exists $x_1, x_2 \in (2, 4), x_1 < x_2$ such that $f'(x_1) = -1$ and $f'(x_2) = 0$.

(S2) there exists $x_3, x_4 \in (2, 4), x_3 < x_4$, such that f is decreasing in $(2, x_4)$, increasing in $(x_4, 4)$ and $2f'(x_3) = \sqrt{3} f(x_4)$.

Then

(1) (S1) is false and (S2) is true
(2) both (S1) and (S2) are true
(3) (S1) is true and (S2) is false
(4) both (S1) and (S2) are false

Ans. (2)

Sol. $f(x) = x^3 - 6x^2 + ax + b$

$f(2) = 8 - 24 + 2a + b = 0$

$2a + b = 16$ ... (1)

$f(4) = 64 - 96 + 4a + b = 0$

$4a + b = 32$ ... (2)

Solving (1) and (2)

$a = 8, b = 0$

$\boxed{f(x) = x^3 - 6x^2 + 8x}$

$f(x) = x^3 - 6x^2 + 8x$

$f'(x) = 3x^2 - 12x + 8$

$f''(x) = 6x - 12$

$\Rightarrow f'(x) \uparrow$ is for $x > 2$, and $f'(x)$ is $\downarrow$ for $x < 2$

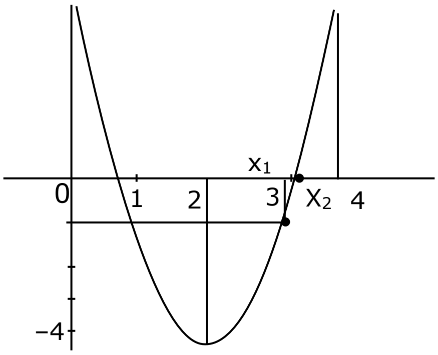
$f'(2) = 12 - 24 + 8 = -4$

$f'(4) = 48 - 48 + 8 = 8$

$f'(x) = 3x^2 - 12x + 8$

vertex (2, – 4)

$f'(2) = -4$, $f'(4) = 8$, $f'(3) = 27 - 36 + 8$



$f'(x_1) = -1$, then $x_1 = 3$

$f'(x_2) = 0$

Again $f'(x) < 0$ for $x \in (2, x_4)$

$f'(x) > 0$ for $x \in (x_4, 4)$

$x_4 \in (3, 4)$

$f(x) = x^3 - 6x^2 + 8x$

$f(3) = 27 - 54 + 24 = -3$

$f(4) = 64 - 96 + 32 = 0$

For $x_4(3, 4)$

$f(x_4) < -3\sqrt{3}$

and $f'(x_3) > -4$

$2f'(x_3) > -8$

So, $2f'(x_3) = \sqrt{3}f(x_4)$

13. Let θ be the acute angle between the tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{1}=1$ and the circle $x^2+y^2=3$ at their point of intersection in the first quadrant. Then $\tan\theta$ is equal to:

(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{4}{\sqrt{3}}$ (3) 2 (4) $\frac{5}{2\sqrt{3}}$

Ans. (1)

Sol. The point of intersection of the curves $\frac{x^2}{9}+\frac{y^2}{1}=1$ and $x^2+y^2=3$ in the first quadrant is $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right)$

Now slope of tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{1}=1$ at $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right)$ is $m_1=-\frac{1}{3\sqrt{3}}$

And slope of tangent to the circle at $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right)$ is $m_2 = -\sqrt{3}$

So, if angle between both curves is θ then $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{-\frac{1}{3\sqrt{3}}+\sqrt{3}}{1\left(-\frac{1}{3\sqrt{3}}(-\sqrt{3})\right)}\right|$

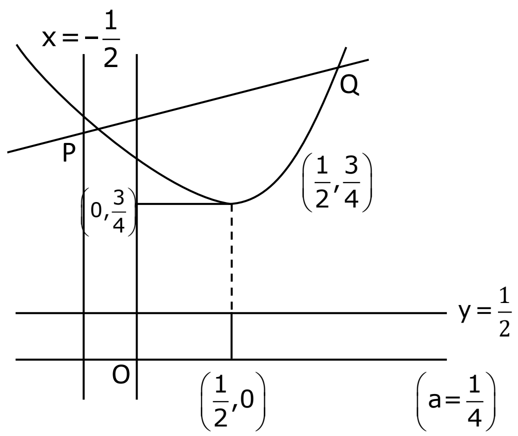
$=\frac{2}{\sqrt{3}}$

14. Consider the parabola with vertex $\left(\frac{1}{2},\frac{3}{4}\right)$ and the directrix $y=\frac{1}{2}$. Let P be the point where the parabola meets the line $x=-\frac{1}{2}$. If the normal to parabola at P intersects the parabola again at the point Q, then $(PQ)^2$ is equal to:

(1) $\frac{25}{2}$ (2) $\frac{75}{8}$ (3) $\frac{15}{2}$ (4) $\frac{125}{16}$

Ans. (4)

Sol.



$\left(y-\frac{3}{4}\right)=\left(x-\frac{1}{2}\right)^2$ ......(1)

For $x=-\frac{1}{2}$

$y - \frac{3}{4} = 1 \Rightarrow y = \frac{7}{4} \Rightarrow P\left(-\frac{1}{2}, \frac{7}{4}\right)$

Now, $y' = 2\left(x - \frac{1}{2}\right)$ At $x = -\frac{1}{2}$.

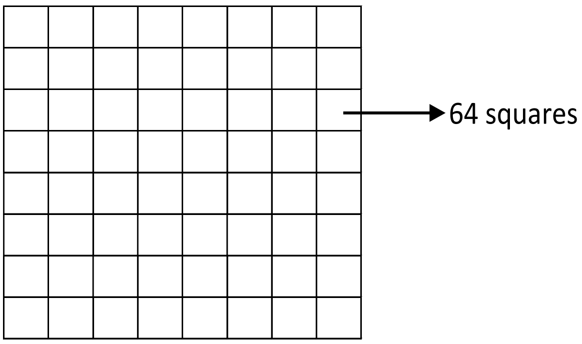
$\frac{x}{2} + 2 - \frac{2}{3} = \left(x - \frac{1}{2}\right)^2$

$\Rightarrow x = 2 \,\&\, -\frac{1}{2}$

$\Rightarrow Q(2,3)$

Now $(PQ)^2 = \frac{125}{16}$

15. Two squares are chosen at random on a chessboard (see figure). The probability that they have a side in common is:



(1) $\frac{1}{18}$ (2) $\frac{1}{7}$ (3) $\frac{1}{9}$ (4) $\frac{2}{7}$

Ans. (1)

Sol. Total ways of choosing square = ${}^{64}C_2$

$= \frac{64 \times 63}{2 \times 1} = 32 \times 63$ ways of choosing two squares having common side

=2(7 x 8)= 112

Required probability = $\frac{112}{32 \times 63} = \frac{16}{32 \times 9} = \frac{1}{18}$.

16. Consider the system of linear equations

$-x + y + 2z = 0$

$3x - ay + 5z = 1$

$2x - 2y - az = 7$

Let S_1 be the set of all $\in$ **R** of for which the system in inconsistent and S_2 be the set of all a $\in$ **R** for which the system has infinitely many solutions. If $n(S_1)$ and $n(S_2)$ denote the number of elements in S_1 and S_2 respectively, than

(1) $n(S_1) = 0, n(S_2) = 2$
(2) $n(S_1) = 2, n(S_2) = 2$
(3) $n(S_1) = 2, n(S_2) = 0$
(4) $n(S_1) = 1, n(S_2) = 0$

Ans. (3)

Sol. $\Delta = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -a & 5 \\ 2 & -2 & -a \end{vmatrix}$

$= -1(a^2 + 10) - 1(-3a - 10) + 2(-6 + 2a)$

$= -a^2 - 10 + 3a + 10 - 12 + 4a$

$\Delta = -a^2 + 7a - 12$

$\Delta = -[a^2 - 7a + 12]$

$\Delta = -[(a - 3)(a - 4)]$

$\Delta_1 = \begin{vmatrix} 0 & 1 & 2 \\ 1 & -a & 5 \\ 7 & -2 & -a \end{vmatrix}$

$= 0 - 1(-a - 35) + 2(-2 + 7a)$

$\Rightarrow a + 35 - 4 + 14a$

$15a + 31$

Now $\qquad \Delta_1 = 15a + 31$

For inconsistent $\Delta = 0 \quad \therefore a = 3, a = 4$

and for a = 3 and 4 $\Delta_1 \neq 0$

$n(S_1) = 2$

For infinite solution $\Delta = 0$

and $\Delta_1 = \Delta_2 = \Delta_3 = 0$

Not possible

$\therefore n(S_2) = 0$

17. Let f: **R** $\to$ **R** be a continuous function. Then $\lim_{x\to \pi/4} \dfrac{\frac{\pi}{4}\int_2^{\sec^2 x} f(x)\,dx}{x^2 - \frac{\pi^2}{16}}$ is equal to:

(1) $4f(2)$ (2) $f(2)$ (3) $2f\left(\sqrt{2}\right)$ (4) $2f(2)$

Ans. (4)

Sol. $\lim_{x\to \pi/4} \dfrac{\frac{\pi}{4}\int_2^{\sec^2 x} f(x)\,dx}{x^2 - \frac{\pi^2}{16}}$

$\lim_{x\to \frac{\pi}{4}} \frac{\pi}{4} \cdot \frac{[f(\sec^2 x).2\sec x.\sec x \tan x]}{2x}$

$\lim_{x\to \frac{\pi}{4}} \frac{\pi}{4}.f(\sec^2 x)\sec^3 x.\frac{\sin x}{x}$

$\frac{\pi}{2}f(2)(\sqrt{2})^3.\frac{1}{\sqrt{2}}.\frac{4}{\pi}$

$\Rightarrow 2f(2)$

18. The range of the function

$f(x) = \log_{\sqrt{5}}\left(3+\cos\left(\frac{3\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{3\pi}{4}-x\right)\right)$ is

(1) $\left[\frac{1}{\sqrt{5}}, \sqrt{5}\right]$ (2) $\left(0, \sqrt{5}\right)$ (3) [0, 2] (4) [–2, 2]

Ans. (3)

Sol. $f(x) = \log_{\sqrt{5}}\left(3+\cos\left(\frac{3\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{3\pi}{4}-x\right)\right)$

$f(x) = \log_{\sqrt{5}}\left[3+2\cos\left(\frac{\pi}{4}\right)\cos(x)-2\sin\left(\frac{3\pi}{4}\right)\sin(x)\right]$

$f(x) = \log_{\sqrt{5}}[3 + \sqrt{2}\,(\cos x - \sin x)]$

Since $-\sqrt{2} \le \cos x - \sin x \le \sqrt{2}$

$\Rightarrow \log_{\sqrt{5}}[3 + \sqrt{2}(-\sqrt{2})] \le f(x) \le \log_{\sqrt{5}}[3 + \sqrt{2}(\sqrt{2})]$

$\Rightarrow \log_{\sqrt{5}}(1) \le f(x) \le \log_{\sqrt{5}}(5)$

So Range of f(x) is [0, 2]

Option (4)

19. Let $J_{n,m} = \int_0^{1/2} \frac{x^n}{x^m - 1} dx, \forall n > m$ and n, m$\in$N. Consider a matrix $A = [a_{ij}]_{3x3}$ where

$a_{ij} = \begin{cases} J_{6+i,3} - J_{i+3,3}, & i \le j \\ 0, & i > j \end{cases}$. Then |adj A^{-1}| is:

(1) $(15)^2 \times 2^{34}$ (2) $(105)^2 \times 2^{38}$ (3) $(15)^2 \times 2^{42}$ (4) $(105)^2 \times 2^{36}$

Ans. (2)

Sol. $\begin{array}{c} \begin{matrix} \surd & \surd & \surd \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$

$J_{6+i,3} - J_{i+3,3}$; $i \le j$

$\Rightarrow \int_0^{1/2} \frac{x^{6+i}}{x^3 - 1} - \int_0^{1/2} \frac{x^{i+3}}{x^3 - 1}$

$\Rightarrow \int_0^{1/2} \frac{x^{i+3}(x^3 - 1)}{x^3 - 1}$

$\Rightarrow \frac{x^{3+i+1}}{3+i+1} = \left(\frac{x^{4+i}}{4+i}\right)^{1/2}$

$a_{ij} = j_{6+i,3} - j_{i+3,3} = \frac{\left(\frac{1}{2}\right)^{4+i}}{4+i}$

$a_{11} = \frac{\left(\frac{1}{2}\right)^5}{5} = \frac{1}{5.2^5}$

$a_{12} = \frac{1}{5.2^5}$

$a_{13} = \frac{1}{5.2^5}$

$a_{22} = \frac{1}{6.2^6}$

$a_{23} = \frac{1}{6.2^6}$

$$a_{33} = \frac{1}{7.2^7}$$

$$A = \begin{bmatrix} \frac{1}{5.2^5} & \frac{1}{5.2^5} & \frac{1}{5.2^5} \\ 0 & \frac{1}{6.2^6} & \frac{1}{6.2^5} \\ 0 & 0 & \frac{1}{7.2^7} \end{bmatrix}$$

$$|A| = \frac{1}{5.2^5}\left[\frac{1}{6.2^6} \times \frac{1}{7.2^7}\right]$$

$$|A| = \frac{1}{210.2^{18}}$$

$$|\text{adj } A^{-1}| = |A^{-1}|^{n-1} = |A^{-1}|^2 = \frac{1}{|A|^2}$$

$\Rightarrow (210.0^{18})^2$

$(105)^2 \times 2^{38}$

20. Let $a_1, a_2, \ldots, a_{21}$ be an AP such that $\sum_{n=1}^{20} \frac{1}{a_n a_{n+1}} = \frac{4}{9\mathbb{R}}$. If the sum of this AP is 189, then $a_6 a_{16}$ is equal to:

(1) 72 (2) 57 (3) 36 (4) 48

Ans. (1)

Sol. $\sum_{n=1}^{20} \frac{1}{a_n a_{n+1}} = \sum_{n=1}^{20} \frac{1}{a_n(a_n + d)}$

$$= \frac{1}{d}\sum_{n=1}^{20}\left(\frac{1}{a_n} - \frac{1}{a_n + d}\right)$$

$$\Rightarrow \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{21}}\right) = \frac{4}{9} \text{ (given)}$$

$$\Rightarrow \frac{1}{d}\left(\frac{a_{21} - a_1}{a_1 a_{21}}\right) = \frac{4}{9}$$

$\Rightarrow \frac{1}{d}\left(\frac{a_1 + 20d - a_1}{a_1 a_2}\right) = \frac{4}{9} \Rightarrow a_1 a_2 = 45$ ...(1)

Now sum of first 21 terms $= \frac{21}{2}(2a_1 + 20d) = 189$

$\Rightarrow a_1 + 10d = 9$ ... (2)

For equation (1) & (2) we get

$a_1 = 3$ & $d = \frac{3}{5}$

OR

$a_1 = 15$ & $d = -\frac{3}{5}$

So, $a_6 \cdot a_{16} = (a_1 + 5d)(a_1 + 15d)$

$\Rightarrow a_6 a_{16} = 72$

Option (2)

Section B

1. All the arrangements, with or without meaning, of the word FARMER are written excluding any word that has two R appearing together. The arrangements are listed serially in the alphabetic order as in the English dictionary. Then the serial number of the word FARMER in this list is _______.

Ans. (77)

Sol. FARMER (6)

A,E,F, M, R, R

A					
E					
F	**A**	**E**			
F	**A**	**M**			
F	**A**	**R**	**E**		
F	**A**	**R**	**M**	**E**	**R**

$\frac{\lfloor 5}{\lfloor 2} - \lfloor 4 = 60 - 24 = 36$

$\frac{\lfloor 3}{\lfloor 2} - \lfloor 2 = 3 - 2 = 1$

= 1

= 2

= 1

77

2. Let $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$. Let a vector $\vec{\upsilon}$ in the plane containing $\vec{a}$ and $\vec{b}$. If $\vec{\upsilon}$ is perpendicular to the vector $3\vec{i} + 2\hat{j} - \hat{k}$ and its projection on $\vec{a}$ is 19 units, then $|2\vec{\upsilon}|^2$ is equal to ________.

Ans. (1494)

Sol. $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}.$

$\vec{c} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\vec{v} = x\vec{a} + y\vec{b}$ $\qquad$ $\vec{v}(3\hat{i} + 2\hat{j} - k) = 0$

$\vec{v}\cdot\hat{a} = 19$

$\vec{v} = \lambda\vec{c} \times (\vec{a} \times \vec{b})$

$\vec{v} = \lambda[(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}]$

$= \lambda[(3+4+1)\left(2\hat{i} - \hat{j} + 2\hat{k}\right) - \left(\frac{6-2-2}{2}\right)\left(\hat{i} + 2\hat{j} + \hat{k}\right)$

$= \lambda[16\hat{i} - 8\hat{j} + 16\hat{k} - 2\hat{i} - 4\hat{j} + 2\hat{k}]$

$\vec{v} = \lambda[14\hat{i} - 12\hat{j} + 18\hat{k}]$

$= \lambda\left[14\hat{i} - 12\hat{j} + 18\hat{k}\right] - \left(\frac{(2\hat{i} - \hat{j} + 2\hat{k})}{\sqrt{4+1+4}}\right) = 19$

$\lambda\frac{[28 + 12 + 36]}{3} = 19$

$\lambda\left(\frac{76}{3}\right) = 19$

$4\lambda = 3 \Rightarrow \lambda = \frac{3}{4}$

$$|2v^2| = \left|2 \times \frac{3}{4}(14\hat{i} - 12\hat{j} + 18\hat{k})\right|^2$$

$$\frac{9}{4} \times 4(7\hat{i} - 6\hat{j} + 9\hat{k})^2$$

$= 9(49 + 36 + 81)$

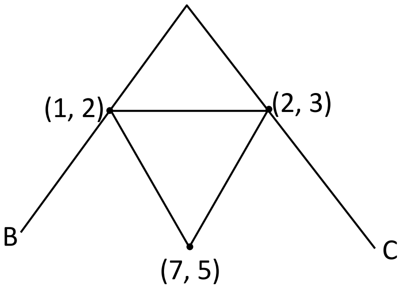
$= 9(166)$

$= 1494$

3. Let the points of intersections of the lines $x-y+1=0$, $x-2y+3=0$ and $2x-5y+11=0$ are the mid points of the sides of a triangle ABC. Then the area of the triangle ABC is ______.

Ans. (6)

Sol. intersection point of give lines are (1, 2), (7, 5), (2,3)



$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 7 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(5-3) - 2(7-2) + 1(21-10)\right]$$

$$= \frac{1}{2}[2 - 10 + 11]$$

$$\Delta DEF = \frac{1}{2}(3) = \frac{3}{2}$$

$$\Delta ABC = 4\Delta DEF = 4\left(\frac{3}{2}\right) = 6$$

4. Let [t] denote the greatest integer $\le$ t. The number of points where the function

$f(x) = [x]\,|x^2 - 1| + \sin\left(\frac{\pi}{[x]+3}\right) - [x+1], x \in (-2, 2)$ is not continuous is ________.

Ans. (2)

Sol. $f(x) = [x]\,|x^2 - 1| + \sin\frac{\pi}{[x+3]} - [x+1]$

$$f(x) = \begin{cases} 3 - 2x^2, & -2 < x < -1 \\ x^2 & -1 \le x < 0 \\ \frac{\sqrt{3}}{2} + 1 & 0 \le x < 1 \\ x^2 + 1 + \frac{1}{\sqrt{2}}, & 1 \le x < 2 \end{cases}$$

discontinuous at x = 0, 1

5. Let X be a random variable with distribution.

x	−2	−1	3	4	6
P(X=x)	$\frac{1}{5}$	a	$\frac{1}{3}$	$\frac{1}{5}$	b

If the mean of X is 2.3 and variance of X σ^2, then 100 σ^2 is equal to:

Ans. (781)

Sol.

x	−2	−1	3	4	6
P(X=x)	$\frac{1}{5}$	a	$\frac{1}{3}$	$\frac{1}{5}$	b

$\overline{X} = 2.3$

$-a + 6b = \frac{9}{10}$ ... (1)

$\sum P_i = \frac{1}{5} + a + \frac{1}{3} + \frac{1}{5} + b = 1$

$a + b = \frac{4}{15}$ ... (2)

From equation (1) and (2)

$a = \frac{1}{10}$, $b = \frac{1}{6}$

$\sigma^2 = \sum p_i x_i^2 - (\overline{X})^2$

$$\frac{1}{5}(4) + a(1) + \frac{1}{3}(9) + \frac{1}{5}(16) + b(36) - (2.3)^2$$

$$= \frac{4}{5} + a + 3 + \frac{16}{5} + 36b - (2.3)^2$$

$$= 4 + a + 3 + 36b - (2.3)^2$$

$$= 7 + a + 36b - (2.3)^2$$

$$= 7 + \frac{1}{10} + 6 - (2.3)^2$$

$$= 13 + \frac{1}{10} - \left(\frac{23}{10}\right)^2$$

$$= \frac{131}{10} - \left(\frac{23}{10}\right)^2$$

$$= \frac{1310 - (23)^2}{100}$$

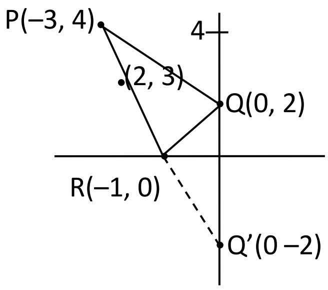
$$\sigma^2 = \frac{781}{100}$$

$$100\sigma^2 = 781$$

6. A man starts walking from the point P(−3, 4), touches the x–axis at R, and then turns to reach at the point Q(0, 2). The man is walking at a constant speed. If the man reaches the point Q in the minimum time, then $50((PR)^2+(RQ)^2)$ is equal to ________.

Ans. (1250)

Sol.



$50(PR^2 + RQ^2)$

$50(20 + 5)$

$50(25)$

$= 1250$

7. If the sum of the coefficients in the expansion of $(x+y)^n$ is 4096, than greatest coefficient in the expansion is_______.

Ans. (924)

Sol. $(x + y)^n \Rightarrow 2n = 4096$

$\Rightarrow 2^n = 2^{12}$

$n = 12$

$2^{10} = 1024 \times 2$

$2^{11} = 2048$

$2^{12} = \underline{4096}$

$${}^{12}C_6 = \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1}$$

$= 11 \times 3 \times 4 \times 7$

$= 924$

8. Let $f(x) = x^6 + 2x^4 + x^3 + 2x + 3, x \in \mathbf{R}$. Then the natural number n for which

$\lim\limits_{x\to1}\frac{x^n f(1) - f(x)}{x-1} = 44$ is ________.

Ans. (7)

Sol. $f(n) = x^6 + 2x^4 + x^3 + 2x + 3$

$$\lim_{x\to1}\frac{x^n f(1) - f(x)}{x-1} = 44$$

$$\lim_{x\to1}\frac{9x^n - (x^6 + 2x^4 + x^3 + 2x + 3)}{x-1} = 44$$

$$\lim_{x\to1}\frac{9nx^{n-1} - (6x^5 + 8x^3 + 3x^2 + 2)}{1} = 44$$

$\Rightarrow 9n - (19) = 44$

$\Rightarrow 9n = 63$

$\Rightarrow n = 7$

9. If for the complex numbers z satisfying $|z-2-2i| \le 1$, the maximum value of $|3iz+6|$ is attained at $a+ib$, then $a+b$ is equal to _________.

Ans. (5)

Sol. $|z - 2 - 2i| \le 1$

$|x + iy - 2 - 2i| \le 1$

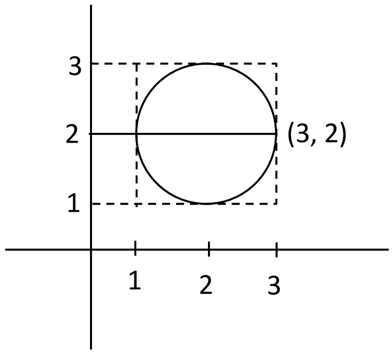
$|(x - 2) + i(y - 2)| \le 1$

$(x - 2)^2 + (y - 2)^2 \le 1$

$|3iz + 6|_{max}$ at a + ib

$| 3i | \left|z + \frac{6}{3i}\right|$

$3 | z - 2i |_{max}$



From Figure maximum distance at 3 + 2i

a + ib = 3 + 2i = a + b = 3 + 2 = 5

10. Let f(x) be a polynomial of degree 3 such that $f(k) = -\frac{2}{k}$ for k=2, 3, 4, 5. Then the value of 52– 10 f(10) is equal to _______.

Ans. (26)

Sol. K f(k) + 2 = λ (x–2) (x – 3) (x – 4) (x – 5) – (1)

put x = 0

we get $\lambda = \frac{1}{60}$

Now put λ in equation (1)

$\Rightarrow kf(k) + 2 = \frac{1}{60}(x - 2)(x - 3)(x - 4)(x - 5)$

Put x = 10

$\Rightarrow 10f(10) + 2 = \frac{1}{60}(8)(7)(6)(5)$

$\Rightarrow 52 - 10f(10) = 52 - 26 = 26$