JEE ADVANCED

ANSWER KEY

2021

MATHEMATICS

Paper-2

QUESTION WITH SOLUTION

MOTION®

SECTION – A

- This section contains SIX (06) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;
  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;
  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
  Zero Marks : 0 If unanswered;
  Negative Marks : −2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
  choosing any other option(s) will get −2 marks.

1. Let

$S_1=\{(i,j,k): i,j,k\in\{1,2,\ldots,10\}\}$,

$S_2=\{(i,j): 1\le i<j+2\le 10,\ i,j\in\{1,2,\ldots,10\}\}$,

$S_3=\{(i,j,k,l): 1\le i<j<k<l,\ i,j,k,l\in\{1,2,\ldots,10\}\}$

and

$S_4=\{(i,j,k,l): i,j,k$ and l are distinct elements in $\{1,2,\ldots,10\}\}$.

If the total number of elements in the set S_r is n_r, $r=1,2,3,4$, then which of the following statements is (are) TRUE ?

(A) $n_1=1000$ (B) $n_2=44$

(C) $n_3=220$ (D) $\frac{n_4}{12}=420$

Ans. A,B,D

$S_1 = \{(i, j, k); i, j, k \in \{1, 2, ...10\}\}$

n_1 = No. of elements is S_1

$n_1 = 10\times10\times10 = 1000$

$n_3 = {}^{10}C_4 \times 1 = 210$; $n_4 = {}^{10}C_4 \times 4! = 5040$

$n_2 \Rightarrow$

	i	j
If	1	8 choices
	2	8 choices
	3	7 choices
	4	6 choices
	5	5 choices
	6	4 choices
	7	3 choices
	8	2 choices
	9	1 choices

$\Rightarrow$ (1 + ...+8) + 8

36 + 8 = 44

2. Consider a triangle PQR having sides of lengths p,q and r opposite to the angles P,Q and R, respectively. Then which of the following statements is (are) TRUE ?

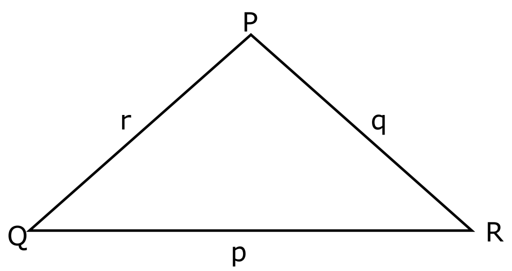
(A) $\cos P \geq 1 - \frac{p^2}{2qr}$

(B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p<q$ and $p<r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

Ans. A,B



(A) $\cos P = \dfrac{q^2 + r^2 - p^2}{2qr} \geq \dfrac{2qr - p^2}{2qr} \geq 1 - \dfrac{p^2}{2qr}$ (by AM$\geq$GM)

$\Rightarrow$ A is correct

(B) $p\cos R + q\cos Q \geq p\cos\theta + q\cos P - r\cos P - r\cos\theta$

pcosR+rcosP+qcosR+rcosQ$\geq$r

q+p$\geq$r

$\Rightarrow$ B is correct

(C) $\dfrac{q+r}{p} \geq \dfrac{2\sqrt{qr}}{p} \geq \dfrac{2\sqrt{\text{SinQ}\sin R}}{\text{SinP}}$ (by AM$\geq$GM)

$\Rightarrow$ C is wrong

(D) In option $\angle R = 90°$ can't be put.

So, D is wrong

3. Let $f:\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\rightarrow\mathbb{R}$ be a continuous function such that

$$f(0)=1 \text{ and } \int_0^{\frac{\pi}{3}} f(t)dt=0$$

Then which of the following statements is (are) TRUE ?

(A) The equation $f(x)-3\cos 3x=0$ has at least one solution in $(0,\dfrac{\pi}{3})$

(B) The equation $f(x)-3\sin 3x=-\dfrac{6}{\pi}$ has at least one solution in $(0,\dfrac{\pi}{3})$

(C) $\lim\limits_{x\to 0}\dfrac{x\int_0^x f(t)\,dt}{1-e^{x^2}} = -1$

(D) $\lim\limits_{x\to 0}\dfrac{\sin x\int_0^x f(t)\,dt}{x^2} = -1$

Ans. **A,B,C**

(A) f(0) = 1 $\displaystyle\int_0^{\frac{\pi}{3}} f(t)\,dt = 0$

(A) $g(x) = \displaystyle\int_0^x f(x)\,dx - \sin 3x$

$\therefore$ By Rolle's Theorem,

g'(x) = 0 at

atleast one value of $x \in (0,\dfrac{\pi}{3})$

(B) $g(x) = \int_0^x f(x)\,dx + \cos 3x + \frac{6}{\pi}x$

$g(0) = 1, \quad g\left(\frac{\pi}{3}\right) = -1 + \frac{6}{\pi}\cdot\frac{\pi}{3} = 1$

$\therefore$ By Rolle's Theorem,

$g'(x) = 0$ at

atleast one value of $x \in (0, \frac{\pi}{3})$

(C) $\lim_{x\to 0} \frac{x\int_0^x f(t)\,dt}{\left(\frac{1-e^{x^2}}{x^2}\right)x^2} = \lim_{x\to 0} \frac{\int_0^x f(t)\,dt}{x}$

$= -\lim_{x\to 0} f(0) = -1$ (by L'hospital rule)

(D) $\lim_{x\to 0} \frac{\int_0^x f(t)\,dt}{x} = \lim_{x\to 0} \frac{f(x)}{1} = 1$ (by L'hospital)

4. For any real numbers α and β, let $y_{\alpha,\beta}(x), x\in\mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx} + \alpha y = xe^{\beta x}, \; y(1) = 1$$

Let $S=\{y_{\alpha,\beta}(x) : \alpha,\beta\in\mathbb{R}\}$. Then which of the following functions belong(s) to the set S ?

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

(B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

Ans. **A,C**

C-1

If $\alpha + \beta \neq 0$

$\frac{dy}{dx} + \alpha y = x.e^{\beta x}$

I.F. $= e^{\alpha x}$

$y.\, e^{\alpha x} = \int x.e^{(\alpha+\beta)}dx$

$$y.e^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \frac{e^{(\alpha+\beta)}x}{(\alpha+\beta)^2} + c$$

If x= 1; y=1

$$e^{\alpha} = \frac{e^{\alpha+\beta}}{\alpha+\beta} - \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2} + c$$

$$e^{\alpha} = e^{\alpha+\beta}\left\{\frac{1}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right\} + c \qquad ....(1)$$

$$Y = \left\{\frac{x}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right\}e^{\beta x}c.e^{-\alpha x}$$

α = 1, β = 1

$$\therefore\ y = \left(\frac{x}{2} - \frac{1}{4}\right)\cdot e^{x} + e^{-x}\left\{e - e^2\left(\frac{1}{2} - \frac{1}{4}\right)\right\}$$

$$= \frac{e^x}{2}\left(x - \frac{1}{2}\right) + e^{-x}\left\{e - \frac{e^2}{4}\right\}$$

$\Rightarrow$ (C)

C-2

If $\alpha+\beta=0$

α = 1 , β= −1

$$\frac{dy}{dx} + y = x.e^{-x}$$

IF = e^x

$$y.e^x = \int x dx$$

$$y.e^x = \frac{x^2}{2} + c$$

x = 1, y =1, c = $e - \frac{1}{2}$

$\Rightarrow$ (A)

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda>0$. If $|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}$, then which of the following statements is (are) TRUE ?

(A) Projection of $\overline{O}\,\overline{C}$ on $\overline{O}\,\overrightarrow{A}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overline{O}\,\overrightarrow{A}$ and $\overline{O}\,\overline{C}$ is $\frac{\pi}{3}$

Ans. A,B,C

$$\overrightarrow{OB}\times\overrightarrow{OC}=\overrightarrow{OB}\times\frac{1}{2}\left(\overrightarrow{OB}-\lambda\overrightarrow{OA}\right)$$

$$=\frac{1}{2}\{\lambda\}\left(\overrightarrow{OA}\times\lambda\overrightarrow{OB}\right)$$

$$=\frac{\lambda}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 2 & 1\\ 1 & -2 & 2\end{vmatrix}$$

$$=\frac{\lambda}{2}\left\{6\hat{i}-3\hat{j}-6\hat{k}\right\}$$

$$\because \frac{\lambda}{2}\sqrt{36+9+36}=\frac{9}{2}$$

$$\Rightarrow \lambda\times 9=9$$

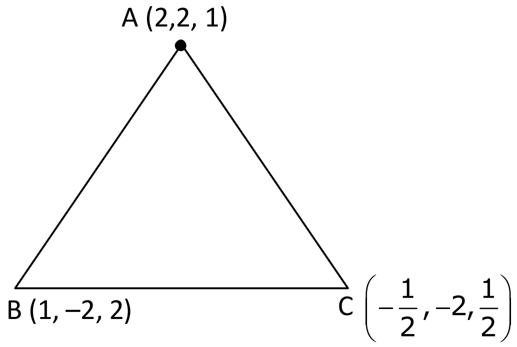
$$\Rightarrow \lambda=1$$

Now,

(A) $$\frac{\overrightarrow{OC}.\overrightarrow{OA}}{\left|\overrightarrow{OA}\right|}=\frac{\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right).\left(\overrightarrow{OA}\right)}{\left|\overrightarrow{OA}\right|}=\frac{\frac{1}{2}\{(2-4+2)-9\}}{3}=\frac{-3}{2}$$

(B) $$\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\left|6\hat{i}-3\hat{j}-6\hat{k}\right|=\frac{9}{2}$$

(C)

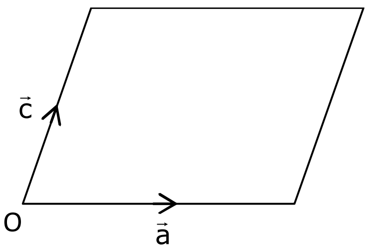


$$\Delta = \frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & 4 & -1 \\ \frac{5}{2} & 4 & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}\left|6\hat{i} - 3\hat{j} - 6\hat{k}\right|$$

$$= \frac{9}{2}$$

(D)



$$\vec{d}_1 = \vec{a} + \vec{c}$$

$$\vec{d}_2 = \vec{a} - \vec{c}$$

$$\therefore\ \vec{d}_1 \cdot d_2 = (\vec{a} + \vec{c}) \cdot (\vec{a} - \vec{c})$$

$$=a^2-c^2$$

$$= a^2 - \left\{\frac{1}{4}(b^2 - a^2)\right\}$$

$$= \frac{5a^2}{4} - \frac{b^2}{4}$$

$$= \frac{5}{4}(a) - \frac{9}{4} = 9$$

$$\because\ 9 = |\vec{a} + \vec{c}|\,|\vec{a} - \vec{c}|\cos\theta$$

$$= \left(a + \frac{b}{a} - \frac{a}{2}\right)\left|a - \frac{b}{2} + \frac{a}{2}\right|\cos\theta$$

$$= \left|\frac{a+b}{2}\right|\left|\frac{3a}{2} - \frac{b}{2}\right|\cos\theta$$

$$= \left|\frac{3,0,3}{2}\right|\left|\frac{5,8,1}{2}\right|\cos\theta$$

$$= \frac{\sqrt{18}}{2} \cdot \frac{\sqrt{25+64+1}}{2}\cos\theta$$

$$9 = \frac{3\sqrt{2}}{2} \times \frac{3\sqrt{10}}{2}\cos\theta \qquad \Rightarrow \cos\theta = \frac{2}{\sqrt{5}}$$

6. Let E denote the parabola $y^2=8x$. Let P $=(-2,4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) TRUE ?

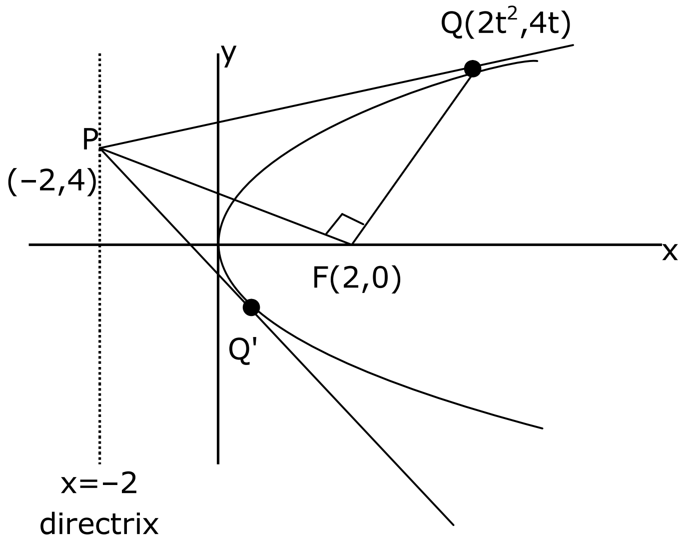
(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Ans. **A,B**



$ty = x + 2t^2$

$4t = -2+2t^2$

$t^2 - 2t - 1 = 0$

$PF = \sqrt{4^2+4^2} = 4\sqrt{2}$, QQ' $\rightarrow$ – focal chord

A,B $\rightarrow$ Highlights of parabola

Section – 2

- This section contains THREE (03) question stems.
- There are TWO (02) questions corresponding to each question stem.
- The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, truncate/round-off the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

  Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

Consider the region R $=\{(x,y)\in\mathbb{R}\times\mathbb{R}: x\geq 0 \text{ and } y^2\leq 4-x\}$. Let $\mathcal{F}$ be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in $\mathcal{F}$. Let (α,β) be a point where the circle C meets the curve $y^2=4-x$.

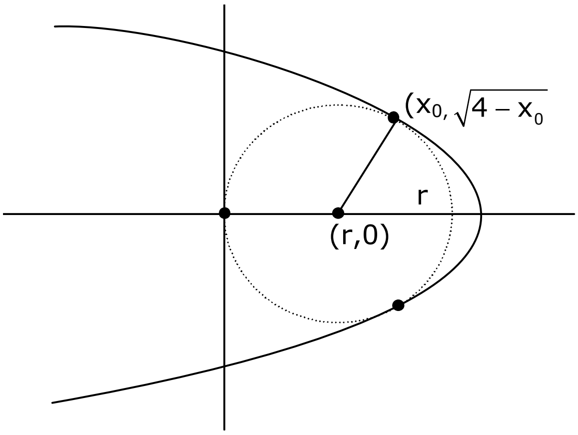
7. The radius of the circle C is ____ .

Ans. **1.5**

8. The value of α is ____ .

Ans. **2**

Sol.



$(x-r)^2 + y^2 = r^2$

$x^2 + r^2 - 2xr + 4-x = r^2$

$$\frac{x^2+4-x}{2x} = r \quad ....(1)$$

$$\therefore \frac{x_0^2+4-x_0}{2x_0} = x_0\frac{-1}{2}$$

$$\frac{x_0^2+4-x_0}{r_0} = 2x_0 - 1$$

$$r_0^2 + 4 - x_0 = 2x_0^2 - x_0$$

$$x_0^2 = 4$$

$$x_0 = 2$$

$$2y\frac{dy}{dx} = -1$$

$$m \in = -\frac{1}{2y_0}$$

$m_n = +2y_0$

$$y - \sqrt{4-x_0} = 2\sqrt{4-x_0}\,(x-x_0)$$

(r,0)

$$-\sqrt{4-x_0} = 2\sqrt{4-x_0}\,(r-x_0)$$

$$r = x_0\frac{-1}{2} \quad .....(2) \qquad \therefore r = \frac{3}{2}$$

Question Stem for Question Nos. 9 and 10

Question Stem

Let $f_1:(0,\infty)\rightarrow\mathbb{R}$ and $f_2:(0,\infty)\rightarrow\mathbb{R}$ be defined by

$$f_1(x)=\int_0^x \prod_{j=1}^{21}(t-j)^j\,dt,\quad x>0$$

and $f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450$, $x>0$,

where, for any positive integer n and real numbers $a_1,a_2,\ldots,a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1,a_2,\ldots,a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i=1,2$, in the interval $(0,\infty)$.

9. The value of $2m_1+3n_1+m_1n_1$ is ____ .

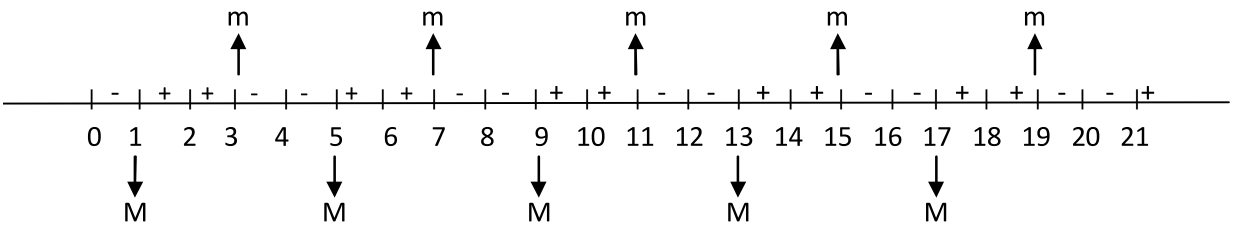
Ans. **57**

10. The value of $6m_2+4n_2+8m_2n_2$ is ____.

Ans. **6**

Sol. $f_1(x)=\int_0^x (t-1)(1-2)^2(t-3)^3\,.......(t-21)^{21}\,dt$

$$\frac{df_1}{dx}=(x-1)(x-2)^2(x-3)^3\,......(x-21)^{21}=0$$

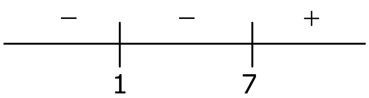


$m_1=6$ $n_1=5$

$$f_2'(x)=98\times 50(x-1)^{49}-600\times 49(x-1)^{48}$$

$$=(x-1)^{48}\{4900(xy)-29400\}$$

$$=(x-1)^{48}\{4900x-34300\}$$



$m_2=1$
$n_2=0$

(9) 12+15+30 = 57.

(10) 6 + 0 = 6.

Question Stem for Question Nos. 11 and 12

Question Stem

Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}, i=1,2$, and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$ be functions such that

$g_1(x)=1$, $g_2(x)=|4x-\pi|$ and $f(x)=\sin^2 x$, for all $x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$

Define

$$S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x).g_i(x)\,dx, i = 1,2$$

11. The value of $\frac{16S_1}{\pi}$ is __________.

Sol. **2**

$$S_1 = \int_{\pi/8}^{3\pi/8} \sin^2 x\ dx$$

$$\frac{1}{2}\int_{\pi/8}^{3\pi/8} (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \left(\frac{\sin 2x}{2}\right)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right]$$

$$= \frac{\pi}{8}$$

$$\Rightarrow \frac{16\times\frac{\pi}{8}}{\pi} = 2$$

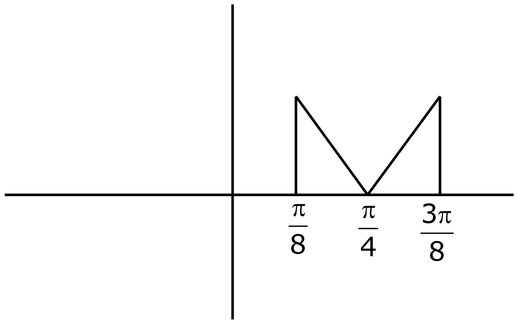
12. The value of $\frac{48S_2}{\pi^2}$ is ________.

Ans. **1.5**

$$S_2 = \int_{\pi/8}^{3\pi/8} \sin^2 |4x-\pi|\,dx$$

$$S_2 = \int_{\pi/8}^{3\pi/8} \cos^2 x\,|2\pi-\pi-4x|\,dx$$

$$\int_{\pi/8}^{3\pi/8} \left(\sin^2 x\,|4x-\pi| + \cos^2 x\,|\pi-4x|\right)dx$$



$$2I = \int_{\pi/8}^{3\pi/8} |4x - \pi|$$

$$2I = \left\{\frac{1}{2} \times \frac{\pi}{8} \times \frac{\pi}{2}\right\}$$

$$2I = \frac{\pi^2}{16}$$

$$S_2 = \frac{\pi^2}{32}$$

$$\therefore \frac{48S_2}{\pi^2} = 1.5$$

Section – 3

- This section contains TWO (02) paragraphs. Based on each paragraph, there are TWO (02) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONLY ONE of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −1 In all other cases.

Paragraph

Let

$$M=\{(x,y)\in\mathbb{R}\times\mathbb{R}: x^2+y^2\le r^2\},$$

where $r>0$. Consider the geometric progression $a_n=\frac{1}{2^{n-1}}$, $n=1,2,3,\dots$. Let $S_0=0$ and, for $n\ge1$, let S_n denote the sum of the first n terms of this progression. For $n\ge1$, let C_n denote the circle with center $(S_{n-1},0)$ and radius a_n, and D_n denote the circle with center (S_{n-1},S_{n-1}) and radius a_n.

13. Consider M with $r=\frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A) $k+2l=22$ (B) $2k+l=26$ (C) $2k+3l=34$ (D) $3k+2l=40$

Ans. **D**

$$S_n = 2\left(1-\frac{1}{2^n}\right)$$

C_1 Centre (0,0), r = 1

C_2 Centre (1,0), r = $\frac{1}{2}$

C_3 Centre $(\frac{3}{2},0)$, r = $\frac{1}{4}$

$x^2 + y^2 = r^2$

$r = \frac{1025}{513}$

$S_{n-1} + r_2 < r_1$

$S_{n-1} + a_n < \frac{1025}{513}$

$S_{n-1} + a_n < \frac{1025}{513}$

$\Rightarrow S_n < \frac{1025}{513} \Rightarrow 1-\frac{1}{2^n} < \frac{1025}{1026}$

$\frac{1}{2^n} > \frac{1}{1026} \Rightarrow 2^n < 1026$

n = 10

$C_1, C_3, C_5, C_7, C_9 \rightarrow$ do not intersect

$C_2, C_4, C_6, C_8, C_{10} \rightarrow$ do not intersect

$\therefore$ maximum 5 circles do not intersect

l = 5

3k + 2l = 40

14. Consider M with $r=\frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199 (C) 200 (D) 201

Ans. **B**

$n = 1 \quad (S_0, S_0) = (0,0)$ centre

$r = 1$

$n = 2 \quad (S_1, S_1) =$ centre $(1,1)$

$r_2 = \frac{1}{2}$

$$\therefore \sqrt{2}S_{n-1} + a < \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$$

$$2\sqrt{2}\left(1-\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}} < \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$$

$$2\sqrt{2}-\frac{\sqrt{2}}{2^{n-2}}+\frac{1}{2^{n-1}} < 2\sqrt{2}-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}}\left(\frac{1}{2}-\sqrt{2}\right) < \frac{-\sqrt{2}}{2^{198}}$$

$$2^{n-2} < \left(2-\frac{1}{\sqrt{2}}\right)^2$$

$\therefore \; n \le 199$

Paragraph

Let $\psi_1:[0,\infty)\to\mathbb{R}$, $\psi_2:[0,\infty)\to\mathbb{R}$, $f:[0,\infty)\to\mathbb{R}$ and $g:[0,\infty)\to\mathbb{R}$ be functions such that $f(0)=g(0)=0$,

$$\psi_1(x)=e^{-x}+x,\ x\ge 0,$$

$$\psi_2(x)=x^2-2x-2e^{-x}+2,\ x\ge 0,$$

$$f(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}\,dt,\ x>0$$

and

$$g(x)=\int_0^{x^2}\sqrt{t}\,e^{-t}\,dt,\ x>0.$$

15. Which of the following statements is TRUE ?

(A) $f\left(\sqrt{\ln 3}\right) + g\left(\sqrt{\ln 3}\right) = \frac{1}{3}$

(B) For every $x>1$, there exists an $\alpha\in(1,x)$ such that $\psi_1(x)=1+\alpha x$

(C) For every $x>0$, there exists a $\beta\in(0,x)$ such that $\psi_2(x)=2x(\psi_1(\beta)-1)$

(D) f is an increasing function on the interval $\left[0,\frac{3}{2}\right]$

Ans. **C**

(A) $f(x) = 2\int_0^x (t - t^2).e^{-t^2}dt$

$g(x) = \int_0^{x^2} \sqrt{t} \cdot e^{-t}dt$ $\qquad t = u^2$

$\qquad dt = 2udu$

$g(x) = 2\int_0^x u^2 \cdot e^{-u^2}du$

$f(x) + g(x) = 2\int_0^x (t - t^2 + t^2)e^{-t^2}dt$

$= 2\int_0^x t \cdot e^{-t^2}dt$ $\qquad t^2 = z$

$\qquad 2tdt = dz$

$= 2\int_0^{x^2} e^{-z}dz \quad = -(e^{-z})^{x^2}$

$= -\{e^{-x^2} - 1\}$

$f(x) + g(x) = 1 - e^{-x^2}$

$\therefore f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = 1 - e^{-\ln 3} = 1 - \frac{1}{3} = \frac{2}{3}$

(B) $e^{-x} + x = 1 + \alpha x$

$e^{-x} - 1 = (\alpha - 1)x$

LHS < 0 and RHS > 0 (not possible)

(C) $\psi_2'(x) = 2x - 2 + 2e^{-x}$

$= 2\psi_1(x) - 2$

$\psi_2'(\beta) = \frac{\psi_2(x) = \psi_2(0)}{x - 0}$

$2\psi_1(\beta) - 2 = \frac{x^2 - 2x - 2e^{-x} + 2}{x}$

$2x(\psi_1(\beta) - 1) = \psi_2(x)$

Hence its true

(D) $f(x) = 2\int_0^x (t - t^2) \cdot e^{-t^2}$

$f'(x) = 2(x - x^2).e^{-x^2}$

$= 2x(1 - x)e^{-x^2}$

(+ve)(+ve)

Changes sign

16. Which of the following statements is TRUE ?

(A) $\psi_1(x) \le 1$, for all $x > 0$

(B) $\psi_2(x) \le 0$, for all $x > 0$

(C) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Ans. **D**

(A) $e^{-x} + x - 1 \le 0 \quad \forall x > 0$

$h(x) = e^{-x} + x - 1$

$h'(x) = -e^{-x} + 1$

$= 1 - \frac{1}{e^x} = \frac{e^x - 1}{e^x}$

$\Rightarrow$ false

(B) $x^2 - 2x - 2e^{-x} + 2 \le 0 \qquad \forall x > 0$

False as $x \to \infty$

LHS $\to \infty$

(C) $f(x) + g(x) = 1 - e^{-x^2}$

$F(x) = 1 - e^{-x^2} - g(x)$ +ve functions

$f(x) =$

$\therefore \left(\frac{2}{3}x^3 - \frac{2}{5}x^5\right) \qquad x \in \left(0, \frac{1}{2}\right)$

$f(x) \le 1 - e^{-x^2} - \left(\frac{2x^3}{3} - \frac{2}{5}x^5\right)$

$g(x) \ge \int_0^{x^2} \sqrt{t}(1 - t)$

$\ge \int_0^{x^2} \left(\sqrt{t} - t^{3/2}\right) dt$

$\ge \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{5}{2}}}{\frac{5}{2}}\right) x^2$

$g(x) \ge \left(\frac{2}{3}x^3 - \frac{2}{5}x^5\right)$

$\Rightarrow$ False

(D) $e^{-t} \le \left(1 - t + \frac{t^2}{2}\right)$

$$\int_0^{x^2} \sqrt{t}e^{-t} \le \int^{x^2} \sqrt{t} \cdot \left(1 - t + \frac{t^2}{2}\right)dt$$

$$g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$$

$\Rightarrow$ true

SECTION 4

- This section contains THREE (03) questions.
- The answer to each question is a NON-NEGATIVE INTEGER.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If ONLY the correct integer is entered;

  Zero Marks : 0 In all other cases.

17. A number is chosen at random from the set {1,2,3,…,2000}. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of 500p is ___ .

Ans. 214

$$\left[\frac{2000}{3}\right] + \left[\frac{2000}{7}\right] - \left[\frac{2000}{21}\right]$$

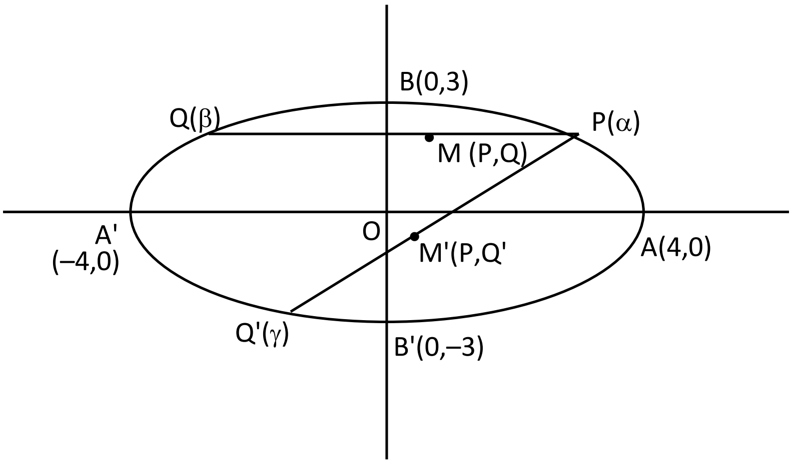
666 + 285 −95

$\therefore$ Prob. = $\frac{856}{2000}$

500P = 214

18. Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P,Q and Q' on E, let M(P, Q) be the mid-point of the line segment joining P and Q, and M(P, Q') be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between M(P, Q) and M(P, Q'), as P, Q and Q' vary on E, is ____.

Ans. 4



$$M\left(\frac{4a\alpha+4a\beta}{2},\frac{4\sin\alpha+3\sin\beta}{2}\right)$$

$$M'\left(\frac{4a\alpha+4a\gamma}{2},\frac{4\sin\alpha+3\sin\gamma}{2}\right)$$

$$MM'=\frac{1}{2}\sqrt{\left(4\cos\beta-4\cos\gamma\right)^2+\left(3\sin\beta-3\sin\gamma\right)^2}$$

$MM'=\frac{1}{2}$ distance between Q & Q'

MM' is free from 'P'

Max of 'QQ' will be possible if QQ' is major axis

$\Rightarrow \quad QQ'=2\times4=8$

$\Rightarrow \quad MM'=\frac{1}{2}\times8=4$

19. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If

$$I=\int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx,$$

then the value of 9I is ______.

Ans. 182

1. $\frac{10x}{x+1}=1$

$10x = x+1$

$x=\frac{1}{9}$

2. $\frac{10x}{x+1}=4$

$10x = 4x + 4$

$6x = 4$

$x=\frac{2}{3}$

3. $\frac{10x}{x+1}=9$ $\qquad 10x = 9x + 9$

$x = 9$

$$\int_0^{\frac{1}{9}} 0dx+\int_{\frac{1}{9}}^{\frac{2}{3}} 1dx+\int_{\frac{2}{3}}^{9} 2dx+\int_{9}^{10} 3dx$$

$$\left(\frac{2}{3}-\frac{1}{9}\right)+2\left(9-\frac{2}{3}\right)+3$$

$$\frac{6-1}{9}+2\left(\frac{25}{3}\right)+3$$

$$\frac{5}{9}+\frac{50}{3}+3$$

$$\left(\frac{5+150+27}{9}\right)\cdot 9$$

$=182$