JEE ADVANCED

ANSWER KEY

2021

MATHEMETICS

Paper-1

QUESTION WITH SOLUTION

MOTION®

SECTION – 1

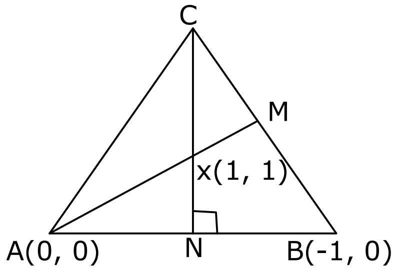
- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −1 In all other cases.

1. Consider a triangle Δ whose two sides lie on the x-axis and the line $x+y+1=0$. If the orthocenter of Δ is (1,1), then the equation of the circle passing through the vertices of the triangle Δ is

(A) $x^2 + y^2 - 3x + y = 0$ (B) $x^2 + y^2 + x + 3y = 0$
(C) $x^2 + y^2 + 2y - 1 = 0$ (D) $x^2 + y^2 + x + y = 0$

Ans. **B**



$L_{Bc} : x + y = 1 \Rightarrow L_{AM} \perp L_{Bc}$

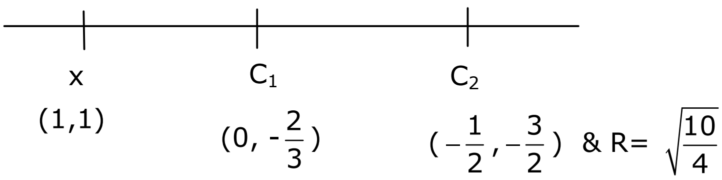
$L_{AM} : y - x = 0$

$\Rightarrow$ A(0,0) & B : (-1,0)

$L_{CN} \perp L_{AB}$

$L_{CN} : x = 1 \Rightarrow$ c: (1,–2)

Now centroid of Δ is $C_1 : (0, -\frac{2}{3})$



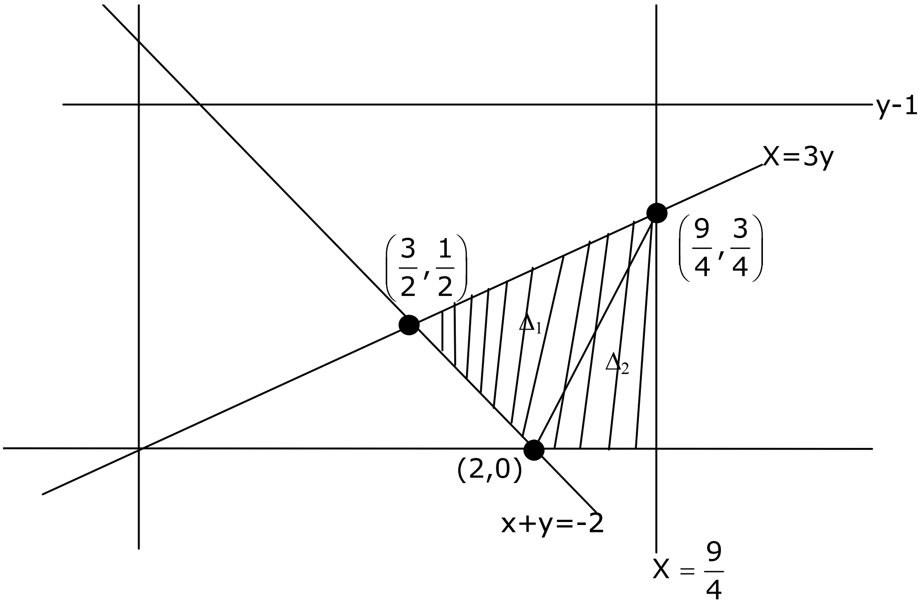
Equation of circle is $\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{3}{2}\right)^2 = \left(\frac{\sqrt{10}}{2}\right)^2$

S: $x^2 + y^2 + x + 3y = 0$

2. The area of the region $\left\{(x, y) : 0 \le x \le \frac{9}{4},\quad 0 \le y \le 1,\quad x \ge 3y,\quad x + y \ge 2\right\}$ is

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$ (C) $\frac{37}{96}$ (D) $\frac{13}{32}$

Ans. **A**



Area = $\Delta_1 + \Delta_2$

$$\Delta = \frac{1}{2}\left|\begin{vmatrix} \frac{3}{2} & \frac{1}{2} & 1 \\ 2 & 0 & 1 \\ \frac{9}{4} & \frac{3}{4} & 1 \end{vmatrix}\right| + \frac{1}{2}\left|\begin{vmatrix} 2 & 0 & 1 \\ \frac{9}{4} & 0 & 1 \\ \frac{9}{4} & \frac{3}{4} & 1 \end{vmatrix}\right|$$

$$\Delta = \frac{1}{2}\left|-2\left(\frac{1}{2} - \frac{3}{4}\right) - 1\left(\frac{3}{2} - \frac{3}{4} - \frac{9}{4} \cdot \frac{1}{2}\right)\right| + \frac{1}{2}\left|\frac{3}{4}\left(2 - \frac{9}{4}\right)\right|$$

$$\Delta = \frac{1}{2}\left|-2\left(-\frac{1}{4}\right) - \left(\frac{9}{8} - \frac{9}{8}\right)\right| + \frac{1}{2}\left|\frac{3}{4}\left(\frac{-1}{4}\right)\right|$$

$$\Delta = \frac{1}{4} + \frac{3}{32} = \frac{8+3}{32} = \frac{11}{32}$$

3. Consider three sets $E_1 = \{1,2,3\}$, $F_1 = \{1,3,4\}$ and $G_1 = \{2,3,4,5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 denote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 denote the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement, from the set G_2 and let S_3 denote the set of these chosen elements.

Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1,2\}$. Then the value of p is

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{2}{5}$

Ans. A

$$P = \frac{P(S_1 \cap (E_1 = E_2))}{P(E_2 = E_3)} = \frac{P(B_{1/2})}{P(B)}$$

$$P(B) = P(B_{1,2}) + P(B_{1,3}) + P(B_{2,3})$$

↑ If 1.2 chosen at start; ↑ If 1.3 chosen at start; ↑ If 2.3 chosen at start

$$P(B_{1,2}) = \frac{1}{3} \times \frac{1 \times {}^2C_1}{{}^3C_2} \times \frac{1}{{}^5C_2}$$

$$P(B_{2,5}) = \frac{1}{3} \times \left[\frac{{}^3C_2 \times 1}{{}^4C_2} \times \frac{1}{{}^4C_2} + \frac{1 \times {}^3C_3}{{}^4C_2} \times \frac{1}{{}^5C_2}\right]$$

$$\frac{P(B_{1,2})}{P(B)} = \frac{1}{5}$$

4. Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \cdots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}, z_k = z_{k-1}e^{i\theta_k}$ for $k = 2,3,\ldots,10$, where $i = \sqrt{-1}$. Consider the statements P and Q given below:

$$P : |z_2 - z_1| + |z_3 - z_2| + \ldots. + |z_{10} - z_9| + |z_1 - z_{10}| \le 2\pi$$

$$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \ldots. + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \le 4\pi$$

Then,

(A) P is **TRUE** and Q is **FALSE**
(B) Q is **TRUE** and P IS **FALSE**
(C) Both P and Q are **TRUE**
(D) Both P and Q are **FALSE**

Ans. C

$z_k = z_{x-1},\ e^{i\theta_k}$

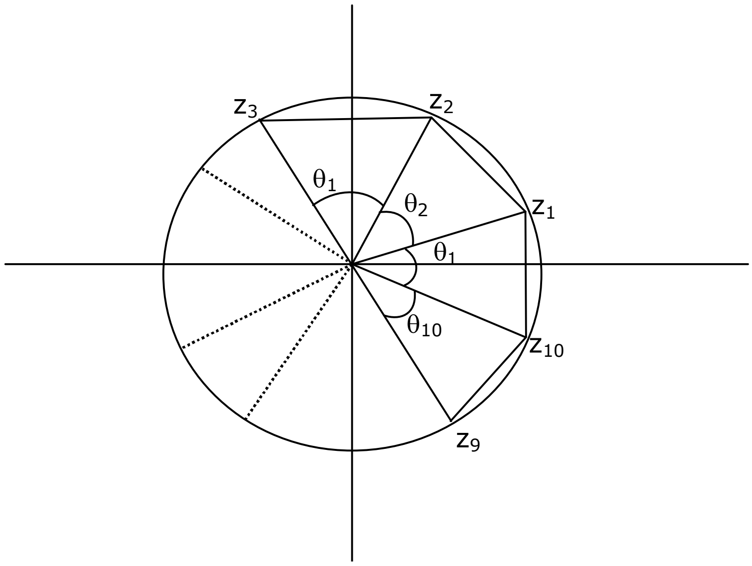
$|z_{k+1} - z_k| =$ side of polygen's

$P = |z_2 - z_1| + |z_3 - z_2| + ...+ |z_1 - z_{10}|$

P = Sum of sides of polygen

$P \le$ pemeter of cirumtance

$\Rightarrow \boxed{P \le 2\pi}$



$\theta : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_1^2 - z_{10}^2|$

$\theta : |z_2 - z_1| \frac{|z_2 + z_1|}{\le 2} + |z_3 - z_2| \frac{|z_3 + z_1|}{\le 2} + ...|z_1 - z_{10}| \frac{|z_1 + z_{10}|}{\le 2}$

$\theta \le 2(|z_2 - z_1| + |z_3 - z_2| + ... + |z_1 - z_{10}|)$

$\theta \le 2.2\pi$

$\boxed{\theta \le 4\pi}$

Section – 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;
  Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 5 and 6

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set $S = \{1,2,3,\ldots,100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

5. The value of $\frac{625}{4}p_1$ is _____.

Ans. 76.25

6. The value of $\frac{125}{4}p_2$ is _____.

Ans. 24.5

P_1=---------------at least 81

$P_1 = 1 - (< 81)$

$$= 1 - \left(\left(\frac{80}{100}\right)^3\right) = 1 - \left(\frac{4}{5}\right)^3 = \frac{125-64}{125} = \frac{61}{125}$$

P_2 = at most 40 is minimum chosen number

$$P_2 = 1 - \left(\frac{60}{100}\right)^3 = 1 - \left(\frac{3}{5}\right)^3 = \frac{125-27}{125} = \frac{98}{125}$$

$$\frac{625}{4}.P_1 = \frac{625}{4}.\frac{61}{125} = \frac{61}{4} = 76.25$$

$$\frac{125}{4}.P_2 = \frac{125}{4}.\frac{98}{125} = 24.5$$

Question Stem for Question Nos. 7 and 8

Question Stem

Let α, β and γ be real numbers such that the system of linear equations

$$x + 2y + 3z = \alpha$$
$$4x + 5y + 6z = \beta$$
$$7x + 8y + 9z = \gamma - 1$$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those $(\alpha,,\gamma)$ for which the above system of linear equations is consistent, and D be the square of the distance of the point (0,1,0) from the plane P.

7. The value of |M| is ____.

Ans. **1**

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

⇒ System of equation has '∞' solution.

$$\Delta_1 = \begin{vmatrix} \alpha & 2 & 3 \\ \beta & 5 & 6 \\ \gamma - 1 & 8 & 9 \end{vmatrix} = 0$$

$-3\alpha + 6\beta - 3(\gamma - 1) = 0$

$-\alpha + 2\beta - \gamma + 1 = 0$

$\alpha - 2\beta + \gamma = 1$ ....(1)

$$\Delta_2 = \begin{vmatrix} 1 & \alpha & 3 \\ 4 & \beta & 6 \\ 7 & \gamma - 1 & 9 \end{vmatrix} = 0$$

$-\alpha(36 - 42) + \beta(9 - 21) - (\gamma - 1)(6 - 12) = 0$

$6\alpha - 12\beta - 6(\gamma - 1) = 0$

$\alpha - 2\beta + \gamma = 1$ ....(2)

$$\Delta_3 = \begin{vmatrix} 1 & 2 & \alpha \\ 4 & 5 & \beta \\ 7 & 8 & \gamma - 1 \end{vmatrix} = 0$$

$\alpha(-3) - \beta(-6) + (\gamma - 1)(-3) = 0$

$\alpha - 2\beta + \gamma = 1$ ....(3)

|M|

$= \alpha - 2(\beta) + \gamma(1)$

$= \alpha - 2\beta + \gamma$

$= 1$

8. The value of D is ____.

Ans. **1.5**

(α, β, γ) lie on plane

$x - 2y + z = 1$ hence its distance

from (0, 1, 0) is

$$D = \left|\frac{-2 - 1}{\sqrt{1 + 4 + 1}}\right| = \frac{3}{\sqrt{6}}$$

$$D^2 = \frac{9}{6} = \frac{3}{2} = 1.5$$

Question Stem for Question Nos. 9 and 10

Question Stem

Consider the lines L_1 and L_2 defined by

$$L_1 : x\sqrt{2} + y - 1 = 0 \text{ and } L_2 : x\sqrt{2} - y + 1 = 0$$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y=2x+1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R′ and S′. Let D be the square of the distance between R′ and S′.

9. The value of λ^2 is _____.

Ans. 9

$d_1.d_2 = \lambda^2$ Let P : (h, k)

$$\left|\frac{h\sqrt{2}+k-1}{\sqrt{3}}\right|\left|\frac{h\sqrt{2}-k+1}{\sqrt{3}}\right| = \lambda^2$$

$$\left|\sqrt{2}x + y - 1\right|\left|\sqrt{2}x - y + 1\right| = 3\lambda^2 \text{ ....(1)}$$

Now solve this with y = 2x + 1 to

Let R & S

$$\left|x\sqrt{2} + 2x + 1 - 1\right|\left|x\sqrt{2} - 2x - 1 + 1\right| = 3\lambda^2$$

$$2|x|^2 = 3\lambda^2$$

$$\lambda^2 = \frac{2x^2}{3}$$

$X_1 = \sqrt{\frac{3}{2}}\lambda \Rightarrow y_1 = \sqrt{6}\lambda + 1$

$X_2 = -\sqrt{\frac{3}{2}}\lambda \Rightarrow y_2 = 1 - \sqrt{6}\lambda$

$$R: \left(\frac{\sqrt{3}}{2}\lambda, \sqrt{6}\lambda + 1\right), S : \left(-\frac{\sqrt{3}}{2}\lambda, 1 - \sqrt{6}\lambda\right)$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 270$$

$$\Rightarrow \left(\sqrt{6}\lambda\right)^2 + \left(2\sqrt{6}\lambda\right)^2 = 270$$

$$6\lambda^2 + 24\lambda^2 = 2 > 0$$

$$\lambda^2 = \frac{270}{30}$$

$$\Rightarrow \boxed{\lambda^2 = 9}$$

10. The value of D is _____.

Ans. **77.14**

$\perp$ bisector of RS

$T = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Here $x_1 + x_2 = 0$

T = (0,1).

Equation of

R'S' " $(y - 1) = -\frac{1}{2}(x - 0) \Rightarrow x + 2y = 2$

$R'(a_1, b)$ $S(a_2, b_2)$

$D = (a_1 - a_2)^2 + (b_1 - b_2)^2 = 5(b_1 - b_2)^2$

Solve $x + 2y = 2$ and $|2x^2 - (y - 1)^2| = 3\lambda^2$

$$\left|8(y-1)^2 - (y-1)^2\right| = 3\lambda^2 \Rightarrow (y-1)^2 = \left(\frac{\sqrt{3}\lambda}{\sqrt{7}}\right)^2$$

$$y - 1 = \pm\frac{\sqrt{3}\lambda}{\sqrt{7}} \Rightarrow b_1 = 1 + \frac{\sqrt{3}\lambda}{\sqrt{7}}, \; b_2 = 1 - \frac{\sqrt{3}\lambda}{\sqrt{17}}$$

$$D = 5\left(\frac{2\sqrt{3}\lambda}{\sqrt{7}}\right)^2 = \frac{5 \times 4 \times 3\lambda^2}{7} = \frac{5 \times 4 \times 27}{7} = 77.14$$

Section – 3

- This section contains SIX (06) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

  Zero Marks : 0 If unanswered;

  Negative Marks : −2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2marks;

choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 mark;
choosing ONLY (B) will get +1 mark;
choosing ONLY (D) will get +1 mark;
choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
choosing any other option(s) will get −2 marks.

11. For any 3 × 3 matrix M, let |M| denote the determinant of M. Let

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix},\ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3 × 3, then which of the following statements is (are) **TRUE**?

(A) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $|(EF)^3| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E = P^{-1}FP$

Ans. A,B,D

$$PEP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 10 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

$|E| = 0$ and $|F| = 0$ and $|Q| \neq 0$

$|EQ| = |E||Q| = 0, |PFQ^{-1}| = \dfrac{|P||F|}{|Q|} = 0$

$T = EQ + PFQ^{-1}$

$TQ = EQ^2 + PF = EQ^2 + P^2EP = EQ^2 + EP = E(Q^2 + P)$

$|TQ| = |E(Q^2 + P)| \Rightarrow |T||Q| = |E||Q^2+P| = 0 \Rightarrow |T| = 0$ (as $|Q| \neq 0$)

(C) $|(EF)^3| > |EF|^2$

Here 0 > 0(false)

(D) as $P^2 = I \Rightarrow P^{-1} = P$ so $P^{-1}FP = PFP = PPEPP = E$

So $E + P^{-1}FP = E + E = 2E$

$P^{-1}EP + F \Rightarrow PEP + F = 2\,PEP$

$Tr\,(2PEP) = 2Tr(PEP) = 2Tr(EPP) = 2Tr(E)$

12. Let $f : R \to R$ be defined by

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$

Then which of the following statements is (are) **TRUE**?

(A) f is decreasing in the interval (–2, –1)

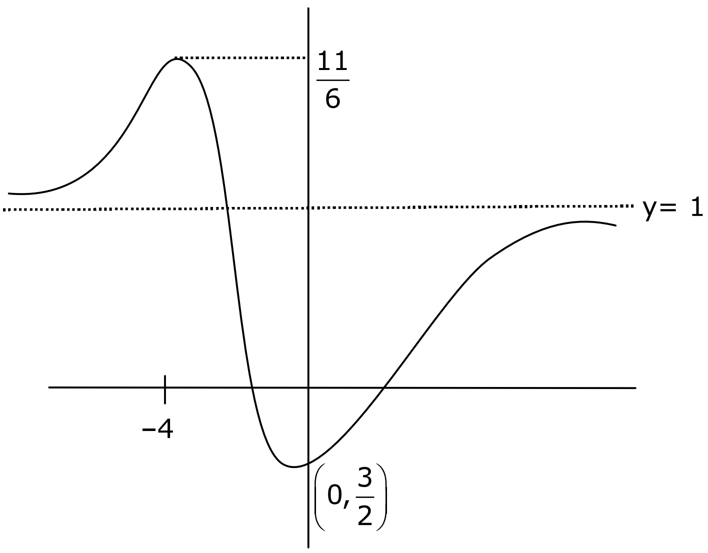
(B) f is increasing in the interval (1, 2)

(C) f is onto

(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

Ans. A,B

$$y = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$



$$\frac{dy}{dx} = \frac{(x^2 + 2x + 4)(2x - 3) - (x^2 - 3x - 6)(2x + 2)}{(x^2 + 2x + 4)^2}$$

$$= \frac{\left[2x^3 + 4x^2 + 8x - 3x^2 - 6x - 12\right] - \left[2x^3 - 6x^2 - 12x + 2x^2 - 6x - 12\right]}{\left[x^2 + 2x + 4\right]^2}$$

$$= \frac{5x^2 + 20x}{\left(x^2 + 2x + 4\right)^2}$$

+ | – | +

–4 0

Increasing in $(-\infty, -4] \cup [0, \infty)$

Decreasing in $\left[-4, 0\right]$

Range $\Rightarrow$ $x^2y + 2xy + 4y = x^2 - 3x - 6$

$\Rightarrow$ $x^2(y - 1) + x(2y + 3) + 4y + 6 = 0$

Case I $\rightarrow$ $y \neq 1$, $D \geq 0$

$(2y + 3)^2 - 4(y - 1)(4y + 6) \geq 0$

$\Rightarrow (4y^2 + 9 + 12) - 4[4y^2 + 2y - 6] \geq 0$

$\Rightarrow -12y^2 + 4y + 33 \geq 0$

$\Rightarrow 12y^2 - 4y - 33 \leq 0$

$\Rightarrow (6y - 11)(2y + 3) \leq 0$

$y \in \left[-\frac{3}{2}, \frac{11}{6}\right] - [1]$

Case–II $\rightarrow$ $y = 1$

$x^2 + 2x + 4 = x^2 - 3x - 6$

$\Rightarrow$ $5x = -10$

$x = -2$ from case –I and case – II

Ans. $\boxed{y \in \left[-\frac{3}{2}, \frac{11}{6}\right]}$

13. Let E, F and G be three events having probabilities

$P(E) = \frac{1}{8}$, $P(F) = \frac{1}{6}$ and $P(G) = \frac{1}{4}$, and let $P(E \cap F \cap G) = \frac{1}{10}$.

For any event H, if H^c denotes its complement, then which of the following statements is (are) **TRUE**?

(A) $P(E \cap F \cap G^c) \leq \frac{1}{40}$

(B) $P(E^c \cap F \cap G) \leq \frac{1}{15}$

(C) $P(E \cap F \cap G) \leq \frac{13}{24}$

(D) $P(E^c \cap F^c \cap G^c) \leq \frac{5}{12}$

Ans. **A,B,C**

$P(6) = \frac{1}{B}$, $P(F) = \frac{1}{6}$, $P(C_1) = \frac{1}{4}$, $P\left(E \cap F \cap G\right) = \frac{1}{10}$

(A) $P(E \cap F \cap G^c) =$ $\quad\quad \therefore \{P(E \cap F) \le P(E)\}$

$\le P(E) - P(G)$

$\le \frac{1}{8} - \frac{1}{10}$

$\le \frac{1}{40}$

(B) $P(E^c \cap F \cap G)$ $\quad\quad \therefore \{P(E \cap F) \le P(E)\}$

$\le P(F) - P(G)$

$\le \frac{1}{6} - \frac{1}{10}$

$\le \frac{10-6}{60} = \frac{4}{60}$

$\le \frac{1}{15}$

(C) $P(E \cup F \cup G) \le P(E) + P(F) + P(G)$

$\le \frac{1}{8} + \frac{1}{6} + \frac{1}{4}$

$\le \frac{15+20+30}{120} = \frac{65}{120} = \frac{13}{24}$

(D) $P(E^c \cap F^c \cap G^c) = 1 - P(E \cup F \cup G)$

$\ge 1 - \frac{13}{24}$

$\ge \frac{11}{24}$

14. For any 3 × 3 matrix M, let |M| denote the determinant of M. Let I be the 3 × 3 identity matrix. Let E and F be two 3 × 3 matrices such that (I – EF) is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE**?

(A) |FE| = |I – FE||FGE| $\quad\quad$ (B) (I – FE) (I + FGE) = I

(C) EFG = GEF $\quad\quad$ (D) (I – FE) (I – FGE) = I

Ans. **A,B,C**

$G = (I - EF)^{-1}$

$G^{-1} = (I - EF)$

$GG^{-1} = G - GEF$

I = G – GEF

Also I = G – EFG

$\Rightarrow$ GEF = EFG $\rightarrow$ $\boxed{c}$

(B) (I – FE)(I + FGE) = I – FE + FGE – FEFGE

= I – FE + FGE – F(G – I)E

= I – FE + FGE – FGE + FE

(D) |FE| = |I − FE| |FGE|

Now (I − FE) (FGE)

= FGE − FEFGE

= FGE − FGEFE

= FGE − F(G − I)E

= FGE − FGE + FE

= FE

⇒ |I − FE||FGE|= |FE|

If B is correct then D is not correct.

15. For any positive integer n, let S_n: $(0, \infty) \to R$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right),$$

where for any $x \in R$, $\cot^{-1}(x) \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE**?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(B) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

Ans. **A,B**

$S_n(x) = \sum_{k=1}^{n} \tan^{-1}\left[\frac{x}{1+(kx)(k+1)x}\right]$

$\sum_{k=1}^{n} \tan^{-1}\left[\frac{(k+1)x - kx}{1+(kx)(k+1)x}\right]$

$\sum_{k=1}^{n}\left[\tan^{-1}((k+1)x) - \tan^{-1}(kx)\right]$

$= \left[\tan^{-1}2x - \tan^{-1}x\right] + \left[\tan^{-1}3x - \tan^{-1}2x\right] + \left[\tan^{-1}4x - \tan^{-1}3x\right] + ... + \left[\tan^{-1}(n+1)x - \tan^{-1}nx\right]$

$S_n(x) = \tan^{-1}(n+1)x - \tan^{-1}x$

Now, $S_{10}(x) = \tan^{-1}x(11x) - \tan^{-1}x$

$= \tan^{-1}\left[\frac{10x}{1+(11x)(x)}\right]$

$= \tan^{-1}\left[\frac{10x}{1+11x^2}\right]$

$= \frac{\pi}{2} - \cot^{-1}\left(\frac{10x}{1+11x^2}\right)$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$ (x > 0)

(b) $\lim_{n\to\infty} \cot\left[\sin(x)\right] = \lim_{n\to\infty} \cot\left(\tan\left[\frac{(n+1)x - x}{1+(n+1)x.x}\right]\right)$

$\lim_{n\to\infty} \cot\left[\cot^{-1}\left(\frac{1+(n+1)x.x}{nx}\right)\right]$

$\lim_{n\to\infty} \frac{1+(n+1)x^2}{(nx)} \Rightarrow x$

$S_n(x) = \tan^{-1}(n+1)x - \tan^{-1}x$

$S_3(x) = \tan^{-1}(4x) - \tan^{-1}x$

$\Rightarrow \tan^{-1}\left[\frac{3x}{1+4x^2}\right] = \frac{\pi}{4}$ (given)

$\Rightarrow \frac{3x}{1+4x^2} = 1$

$\Rightarrow 1 + 4x^2 = 3x$

$\Rightarrow 4x^2 - 3x + 1 = 0$

$D = 9 - 16 < 0$

No real roots

(d) $\tan\left(S_n(x)\right) \le \frac{1}{2}$

$\tan\left[\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)\right] = \frac{nx}{1+(n+1)x^2}$

$\lim_{n\to\infty} \frac{nx}{1+(n+1)x^2} = \frac{1}{x}$

16. For any complex number w = c + id, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers z = x + iy satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0$$

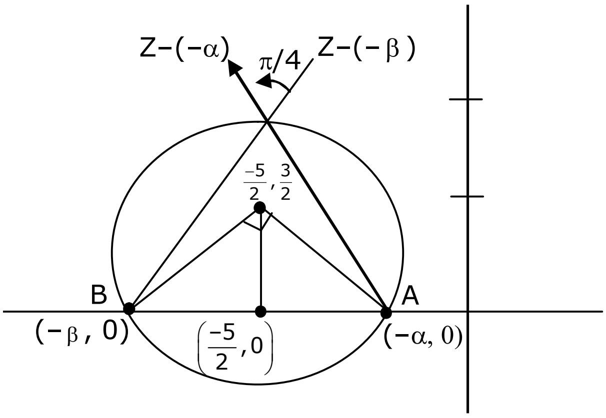
Then which of the following statements is (are) **TRUE** ?

(A) $\alpha = -1$ (B) $\alpha\beta = 4$ (C) $\alpha\beta = -4$ (D) $\beta = 4$

Ans. **B,D**

S : $x^2\ y^2 + 5x - 3y + 4 = 0$

C : $\left(\frac{-5}{2}, \frac{3}{2}\right)$ R = $\sqrt{\frac{25}{4} + \frac{9}{4} - 4} = \frac{3}{\sqrt{12}}$



$$\arg\left(\frac{z-(-\alpha)}{z-(-\beta)}\right) = \frac{\pi}{4}$$

⇒ A : (–1, 0) & B : (–4,0)

$\alpha = 1$, & $\beta = 4$

Alternet:

Let Z = x + iy

$$\tan^{-1}\left[\frac{y}{x+\alpha}\right] - \tan^{-1}\left[\frac{y}{x+\beta}\right] = \frac{\pi}{4}$$

$$\tan^{-1}\left[\frac{\frac{y}{x+\alpha} - \frac{y}{x+\beta}}{1 + \frac{y^2}{(x+\alpha)(x+\beta)}}\right] = \frac{\pi}{4}$$

$$\frac{\frac{y(\beta-\alpha)}{(x+\alpha)(x+\beta)}}{\frac{(x+\alpha)(x+\beta)+y^2}{(x+\alpha)(x+\beta)}} = 1$$

$$y(\beta-\alpha) = x^2 + (\alpha+\beta)x + \alpha\beta + y^2$$

$$\Rightarrow x^2 + y^2 + (\alpha+\beta)x - (\beta-\alpha)y + \alpha\beta = 0$$

$$\left.\begin{array}{l}\alpha+\beta = 5 \\ \alpha-\beta = -3\end{array}\right] \quad \alpha\beta = 4$$

$\alpha = 1$

$\beta = 4$

SECTION 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered;

Zero Marks : 0 In all other cases.

17. For $x \in R$, the number of real roots of the equation $3x^2 - 4|x^2 - 1| + x - 1 = 0$ is ______.

Ans. **4**

$3x^2 - 4|x^2 - 1| + x - 1 = 0$

$3x^2 + x - 1 = 4|x^2 - 1|$

$x^2 \geq 1 \Rightarrow 3x^2 + x - 1 = 4 \times 2 - 4$

$X^2 - x - 3 = 0$ → $x = \dfrac{1+\sqrt{13}}{2}$, $x = \dfrac{1-\sqrt{3}}{2}$

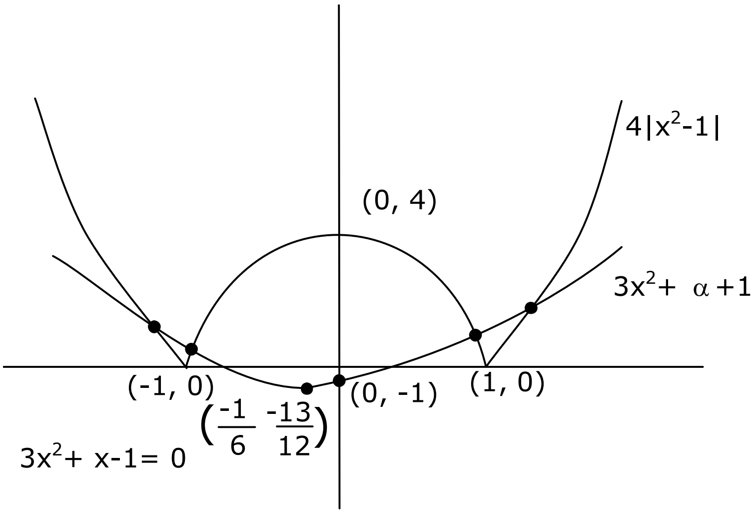
$\Rightarrow 3x^2 + x - 1 = 4 - 4x^2$

→ $x = \dfrac{-1+\sqrt{141}}{14}$, $x = \dfrac{-1-\sqrt{141}}{14}$

$7x^2 + x - 5 = 0$

$3x^2 + x - 1 = 4|x^2 - 1|$

Alternet method



18. In a triangle ABC, let AB = $\sqrt{23}$, BC = 3 and CA = 4. Then the value of $\frac{\cot A + \cot C}{\cot B}$ is ____.

Ans. **2**

AB = c = $\sqrt{23}$

BC = a = 3

CA = b = 4

$\cot A = \frac{b^2 + c^2 - a^2}{2bcKa}$

$\cot B = \frac{c^2 + a^2 - b^2}{2cakb}$

$\cot C = \frac{a^2 + b^2 - c^2}{2abkc}$

$\frac{\cot A + \cot C}{\cot B}$

$\frac{b^2 + c^2 - a^2 + a^2 + b^2 - c^2}{c^2 + a^2 - b^2}$

$= \frac{2.16}{23 + 9 - 16}$

$= \frac{32}{16} = \boxed{2}$

19. Let $\vec{u}, \vec{v}$ and $\vec{w}$, be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and $\vec{u}.\vec{w} = 1$, $\vec{v}.\vec{w} = 1$, $\vec{w}.\vec{w} = 4$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, is $\sqrt{2}$ then the value of $|3\vec{u} + 5\vec{v}|$ is _____.

Ans. **7**

$\bar{u}.\bar{u} = 1$, $\bar{u}.\bar{w} = 1$, $|\bar{w}| = 2$

$\bar{v}.\bar{v} = 1$, $\bar{v}.\bar{w} = 1$

$\bar{u}.\bar{v} \neq 0$

Volume of parallelopiped = $[\bar{u}.\bar{v}.\bar{w}] = \sqrt{2}$

$$\left(\sqrt{2}\right)^2 = \begin{vmatrix} \bar{u}.\bar{u} & \bar{u}.\bar{v} & \bar{u}.\bar{w} \\ \bar{v}.\bar{u} & \bar{v}.\bar{v} & \bar{v}.\bar{w} \\ \bar{w}.\bar{u} & \bar{w}.\bar{v} & \bar{w}.\bar{w} \end{vmatrix}$$

Let $\bar{u}.\bar{v} = \lambda \neq 0$

$$2 = \begin{vmatrix} 1 & \lambda & 1 \\ \lambda & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$2 = (4 - 1) - \lambda(4\lambda - 1) + 1\ (\lambda - 1)$

$2 = 3 - 4\lambda^2 + \lambda + \lambda - 1$

$4\lambda^2 - 2\lambda = 0$

$\lambda = 0$ (not possible)

$\lambda = \frac{1}{2}$

Now $|3\bar{u} + 5\bar{v}| = \sqrt{9 + 25 + 30.\frac{1}{2}}$

$= \sqrt{9 + 25 + 15}$

$= \sqrt{9 + 40}$

$= \sqrt{49} = 7$