JEE ADVANCED

ANSWER KEY

2021

MATHEMATICS

Paper-2

QUESTION WITH SOLUTION

SECTION – A

- This section contains SIX (06) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;
  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;
  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
  Zero Marks : 0 If unanswered;
  Negative Marks : −2 In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
  choosing any other option(s) will get −2 marks.

522316 **Permutation Combination**

1. Let

$S_1=\{(i,j,k): i,j,k\in\{1,2,\ldots,10\}\}$,

$S_2=\{(i,j): 1\le i<j+2\le 10,\ i,j\in\{1,2,\ldots,10\}\}$,

$S_3=\{(i,j,k,l): 1\le i<j<k<l,\ i,j,k,l\in\{1,2,\ldots,10\}\}$

and

$S_4=\{(i,j,k,l): i,j,k$ and l are distinct elements in $\{1,2,\ldots,10\}\}$.

If the total number of elements in the set S_r is n_r, $r=1,2,3,4$, then which of the following statements is (are) TRUE ?

(A) $n_1=1000$ (B) $n_2=44$

(C) $n_3=220$ (D) $\frac{n_4}{12}=420$

माना

$S_1=\{(i,j,k): i,j,k\in\{1,2,\ldots,10\}\}$,

$S_2=\{(i,j): 1\le i<j+2\le 10,\ i,j\in\{1,2,\ldots,10\}\}$,

$S_3=\{(i,j,k,l): 1\le i<j<k<l,\ i,j,k,l\in\{1,2,\ldots,10\}\}$

तथा

$S_4=\{(i,j,k,l): i,j,k$ तथा $l,\ \{1,2,\ldots,10\}\}$ में भिन्न अवयव है।

यदि समुच्चय S_r में अवयवों की कुल संख्या n_r, $r=1,2,3,4$, है, तब निम्न में से कौनसा/कौनसे कथन सत्य है/हैं?

(A) $n_1=1000$ (B) $n_2=44$

(C) $n_3=220$ (D) $\frac{n_4}{12}=420$

Ans. A,B,D

$S_1 = \{(i,j,k); i,j,k \in \{1,2,.....10\}\}$

n_1 = No. of elements in S_1

$n_1 = 10\times10\times10 = 1000$

$n_3 = {}^{10}C_4 \times 1 = 210$; $n_4 = {}^{10}C_4 \times 4! = 5040$

$n_2 \Rightarrow$

	i	j
If	1	8 choices
	2	8 choices
	3	7 choices
	4	6 choices
	5	5 choices
	6	4 choices
	7	3 choices
	8	2 choices
	9	1 choices

$\Rightarrow (1 + \ldots\ldots+8) + 8$

$= 36 + 8 = 44$

522317 **Solution of Triangle**

2. Consider a triangle PQR having sides of lengths p,q and r opposite to the angles P,Q and R, respectively. Then which of the following statements is (are) TRUE ?

(A) $\cos P \ge 1-\frac{p^2}{2qr}$ (B) $\cos R \ge \left(\frac{q-r}{p+q}\right)\cos P+\left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p}<2\frac{\sqrt{\sin Q \sin R}}{\sin P}$ (D) If $<q$ and $p<r$, then $\cos Q>\frac{p}{r}$ and $\cos R>\frac{p}{q}$

माना एक त्रिभुज PQR है, जिसमें कोणों P,Q तथा R के सम्मुख भुजाओं की लम्बाईयाँ क्रमशः p,q तथा r है। तब निम्नलिखित कथनों में से कौनसा/कौनसे सत्य है/हैं?

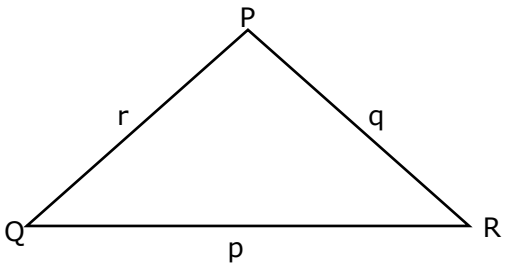
(A) $\cos P \geq 1-\frac{p^2}{2qr}$

(B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P+\left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p}<2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) यदि $p<q$ तथा $p<r$, तब $\cos Q>\frac{p}{r}$ तथा $\cos R>\frac{p}{q}$ है

Ans. A,B



(A) $\cos P=\frac{q^2+r^2-p^2}{2qr} \geq \frac{2qr-p^2}{2qr} \geq 1-\frac{p^2}{2qr}$ (by AM $\geq$ GM)

$\Rightarrow$ A is correct

(B) $p\cos R+q\cos Q \geq p\cos\theta+q\cos P-r\cos P-r\cos\theta$

$p\cos R+r\cos P+q\cos R+r\cos Q \geq r$

$q+p \geq r$

$\Rightarrow$ B is correct

(C) $\frac{q+r}{p} \geq \frac{2\sqrt{qr}}{p} \geq \frac{2\sqrt{\text{SinQ sinR}}}{\text{SinP}}$ (by AM $\geq$ GM)

$\Rightarrow$ C is wrong

(D) In option $\angle R=90^\circ$ can't be put.

So, D is wrong

522320 **Definite Integral**

3. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that

$$(0)=1 \text{ and } \int_0^{\frac{\pi}{3}} (t)dt=0$$

Then which of the following statements is (are) TRUE ?

(A) The equation $(x)-3\cos 3x=0$ has at least one solution in $(0,\frac{\pi}{3})$

(B) The equation $(x)-3\sin 3x=-\frac{6}{\pi}$ has at least one solution in $(0,\frac{\pi}{3})$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

माना f: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ एक सतत् फलन है जबकि

$f(0)=1$ तथा $\int_0^{\frac{\pi}{3}} f(t)dt = 0$

तब निम्न में से कौनसा/कौनसे कथन सही है/हैं?

(A) $(0, \frac{\pi}{3})$ में समीकरण $f(x) - 3\cos 3x = 0$ का कम से कम एक हल है

(B) $(0, \frac{\pi}{3})$ में समीकरण $f(x) - 3\sin 3x = -\frac{6}{\pi}$ का कम से कम एक हल है

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)\,dt}{1-e^{x^2}} = -1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)\,dt}{x^2} = -1$

Ans. A,B,C

$f(0) = 1, \quad \int_0^{\frac{\pi}{3}} f(t)\,dt = 0$

(A) $g(x) = \int_0^x f(x)\,dx - \sin 3x$

$g(0) = 0, \; g\left(\frac{\pi}{3}\right) = 0$

$\therefore$ By Rolle's Theorem,
g'(x) = 0 at

atleast one value of $x \in (0, \frac{\pi}{3})$

(B) $g(x) = \int_0^x f(x)\,dx + \cos 3x + \frac{6}{\pi}x$

$g(0) = 1, \quad g\left(\frac{\pi}{3}\right) = -1 + \frac{6}{\pi}\cdot\frac{\pi}{3} = 1$

$\therefore$ By Rolle's Theorem,
g'(x) = 0 at

atleast one value of $x \in (0, \frac{\pi}{3})$

(C) $$\lim_{x\to 0}\frac{\int_0^x f(t)\,dt}{\left(\frac{1-e^{x^2}}{x^2}\right)x^2} = -\lim_{x\to 0}\frac{\int_0^x f(t)\,dt}{x}$$

$= -\lim_{x\to 0} f(0) = -1$ (by L'hospital rule)

(D) $$\lim_{x\to 0}\frac{\int_0^x f(t)\,dt}{x} = \lim_{x\to 0}\frac{f(x)}{1} = 1$$ (by L'hospital)

522451 **Differential equation**

4. For any real numbers α and β, let $y_{\alpha}(x), x\in\mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx}+\alpha y = xe^{\beta x},\ y(1) = 1$$

Let $S=\{y_{\alpha,\beta}(x):\alpha,\beta\in\mathbb{R}\}$. Then which of the following functions belong(s) to the set S ?

(A) $f(x)=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$ (B) $f(x)=-\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$

(C) $f(x)=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{4}\right)e^{-x}$ (D) $f(x)=\frac{e^x}{2}\left(\frac{1}{2}-x\right)+\left(e+\frac{e^2}{4}\right)e^{-x}$

किन्हीं वास्तविक संख्याओं α तथा β के लिए, माना $y_{\alpha}(x), x\in\mathbb{R}$, अवकलन समीकरण

$\frac{dy}{dx}+\alpha y = xe^{\beta x},\ y(1) = 1$ का एक हल है।

माना $S=\{y_{\alpha,\beta}(x):\alpha,\beta\in\mathbb{R}\}$ है। तब निम्नलिखित फलनों में से कौनसा समुच्चय S से संबंधित है/हैं ?

(A) $f(x)=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$ (B) $f(x)=-\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$

(C) $f(x)=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{4}\right)e^{-x}$ (D) $f(x)=\frac{e^x}{2}\left(\frac{1}{2}-x\right)+\left(e+\frac{e^2}{4}\right)e^{-x}$

Ans. **A,C**

C-1

If $\alpha+\beta\neq 0$

$\frac{dy}{dx}+\alpha y = x.e^{\beta x}$

I.F. $= e^{\alpha x}$

$y.\ e^{\alpha x} = \int x.e^{(\alpha+\beta)}dx$

$$y.e^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \frac{e^{(\alpha+\beta)}x}{(\alpha+\beta)^2} + c$$

If x= 1; y=1

$$e^{\alpha} = \frac{e^{\alpha+\beta}}{\alpha+\beta} - \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2} + c$$

$$e^{\alpha} = e^{\alpha+\beta}\left\{\frac{1}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right\} + c \qquad ....(1)$$

$$Y = \left\{\frac{x}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right\} e^{\beta x} c.e^{-\alpha x}$$

$\alpha = 1, \beta = 1$

$$\therefore\ y = \left(\frac{x}{2} - \frac{1}{4}\right)\cdot e^{x} + e^{-x}\left\{e - e^{2}\left(\frac{1}{2} - \frac{1}{4}\right)\right\}$$

$$= \frac{e^{x}}{2}\left(x - \frac{1}{2}\right) + e^{-x}\left\{e - \frac{e^{2}}{4}\right\}$$

$\Rightarrow (C)$

C-2

If $\alpha + \beta = 0$

$\alpha = 1$, $\beta = -1$

$$\frac{dy}{dx} + y = x.e^{-x}$$

IF $= e^{x}$

$$y.e^{x} = \int x dx$$

$$y.e^{x} = \frac{x^2}{2} + c$$

$$x = 1,\ y = 1,\ c = e - \frac{1}{2}$$

$\Rightarrow$ (A)

522474 **Vector**

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If $|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}$, then which of the following statements is (are) TRUE ?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

माना O मूलबिंदु है तथा $\lambda > 0$ के लिए $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ तथा $\overrightarrow{OC}$ $\frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ यदि $|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}$, तब निम्न में से कौनसा/कौनसे कथन सही है/हैं?

(A) $\overrightarrow{OC}$ का $\overrightarrow{OA}$ पर प्रक्षेप $-\frac{3}{2}$ है

(B) त्रिभुज OAB का क्षेत्रफल $\frac{9}{2}$ है

(C) त्रिभुज ABC का क्षेत्रफल $\frac{9}{2}$ है

(D) समान्तर चतुर्भुज जिसकी आसन्न भुजाएँ $\overrightarrow{OA}$ तथा $\overrightarrow{OC}$ है, के विकर्णो के मध्य न्यून कोण $\frac{\pi}{3}$ है।

Ans. **A,B,C**

$$\overrightarrow{OB} \times \overrightarrow{OC} = \overrightarrow{OB} \times \frac{1}{2}\left(\overrightarrow{OB} - \lambda\overrightarrow{OA}\right)$$

$$= \frac{1}{2}\{\lambda\}\left(\overrightarrow{OA} \times \lambda\overrightarrow{OB}\right)$$

$$= \frac{\lambda}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \frac{\lambda}{2}\left\{6\hat{i} - 3\hat{j} - 6\hat{k}\right\}$$

$$\because \frac{\lambda}{2}\sqrt{36 + 9 + 36} = \frac{9}{2}$$

$$\Rightarrow \lambda \times 9 = 9$$

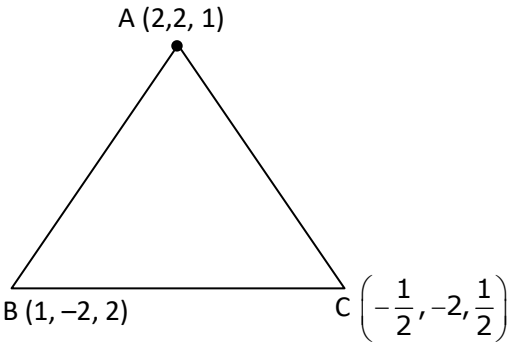
$\Rightarrow \lambda = 1$

Now,

(A) $\dfrac{\overrightarrow{OC}.\overrightarrow{OA}}{\left|\overrightarrow{OA}\right|} = \dfrac{\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right).\left(\overrightarrow{OA}\right)}{\left|\overrightarrow{OA}\right|} = \dfrac{\frac{1}{2}\{(2-4+2)-9\}}{3} = \dfrac{-3}{2}$

(B) $\dfrac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right| = \dfrac{1}{2}\left|6\hat{i}-3\hat{j}-6\hat{k}\right| = \dfrac{9}{2}$

(C)

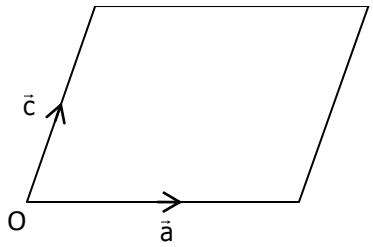


$$\Delta = \frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & 4 & -1 \\ \frac{5}{2} & 4 & \frac{1}{2} \end{vmatrix}$$

$= \dfrac{1}{2}\left|6\hat{i}-3\hat{j}-6\hat{k}\right|$

$= \dfrac{9}{2}$

(D)



$\vec{d}_1 = \vec{a}+\vec{c}$

$\vec{d}_2 = \vec{a}-\vec{c}$

$\therefore \vec{d}_1 \cdot d_2 = (\vec{a}+\vec{c})\cdot(\vec{a}-\vec{c})$

$=a^2-c^2$

$= a^2 - \left\{\dfrac{1}{4}\left(b^2-a^2\right)\right\}$

$$= \frac{5a^2}{4} - \frac{b^2}{4}$$

$$= \frac{5}{4}(a) - \frac{9}{4} = 9$$

$$\because\ 9 = |\vec{a}+\vec{c}||\vec{a}-\vec{c}|\cos\theta$$

$$= \left(a + \frac{b}{a} - \frac{a}{2}\right)\left|a - \frac{b}{2} + \frac{a}{2}\right|\cos\theta$$

$$= \left|\frac{a+b}{2}\right|\left|\frac{3a}{2} - \frac{b}{2}\right|\cos\theta$$

$$= \left|\frac{3,0,3}{2}\right|\left|\frac{5,8,1}{2}\right|\cos\theta$$

$$= \frac{\sqrt{18}}{2}\cdot\frac{\sqrt{25+64+1}}{2}\cos\theta$$

$$9 = \frac{3\sqrt{2}}{2}\times\frac{3\sqrt{10}}{2}\cos\theta \quad \Rightarrow \cos\theta = \frac{2}{\sqrt{5}}$$

522499 **Parabola, Ellipse & Hyperbola**

6. Let E denote the parabola $y^2=8x$. Let P =(−2,4), and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) TRUE ?

(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

माना E परवलय $y^2 = 8x$ को निरूपित करता है। माना P = (−2, 4) तथा Q व Q',E पर दो भिन्न बिंदु इस प्रकार है कि रेखाएँ PQ व PQ', E पर स्पर्श रेखाएँ है। माना F,E की नाभि है। तब निम्न में से कौनसा/कौनसे कथन सही है/हैं ?

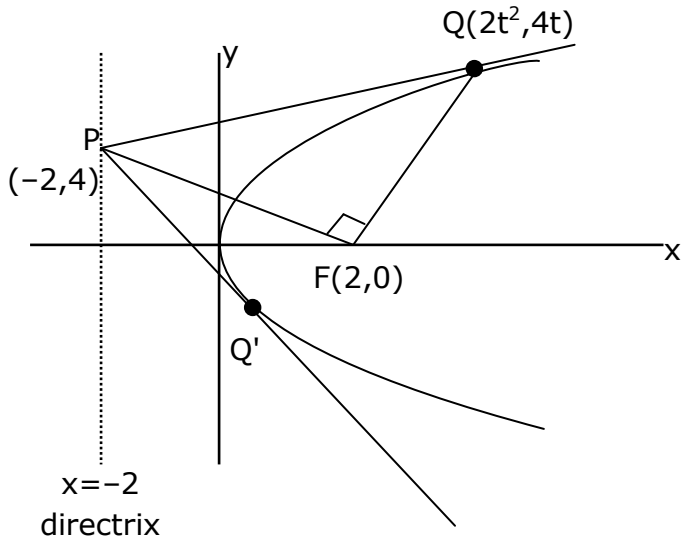
(A) त्रिभुज PFQ एक समकोणीय त्रिभुज है

(B) त्रिभुज QPQ' एक समकोणीय त्रिभुज है

(C) P व F के मध्य दूरी है

(D) F,Q व Q' को मिलाने वाली रेखा पर स्थित है

Ans. A,B



$ty = x + 2t^2$

$4t = -2+2t^2$

$t^2 - 2t - 1 = 0$

$PF = \sqrt{4^2 + 4^2} = 4\sqrt{2}$, QQ' $\rightarrow$ – focal chord

A,B $\rightarrow$ Highlights of parabola

Section – 2

- This section contains THREE (03) question stems.
- There are TWO (02) questions corresponding to each question stem.
- The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, truncate/round-off the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

  Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

Consider the region R $=\{(x,y)\in\mathbb{R}\times\mathbb{R}: x\geq 0 \text{ and } y^2\leq 4-x\}$. Let $\mathcal{F}$ be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in $\mathcal{F}$. Let $(\alpha,)$ be a point where the circle C meets the curve $y^2=4-x$.

प्रश्न संख्या 22 तथा 23 के लिए अनुच्छेद

क्षेत्र R $=\{(x,y)\in\mathbb{R}\times\mathbb{R}: x\geq 0$ तथा $y^2\leq 4-x\}$ पर विचार कीजिए। माना F उन सभी वृत्तों का कुल है जो R में स्थित है तथा जिनके केन्द्र x-अक्ष पर है। माना C वृत्त है जिसकी F के वृत्तों में से सबसे बड़ी त्रिज्या है। माना (α,β) एक बिन्दु है जहाँ वृत्त C वक्र $y^2 = 4-x$ से मिलता है।

522508 **Parabola, Ellipse & Hyperbola**

7. The radius of the circle C is ___ .
वृत्त C की त्रिज्या _____ है।

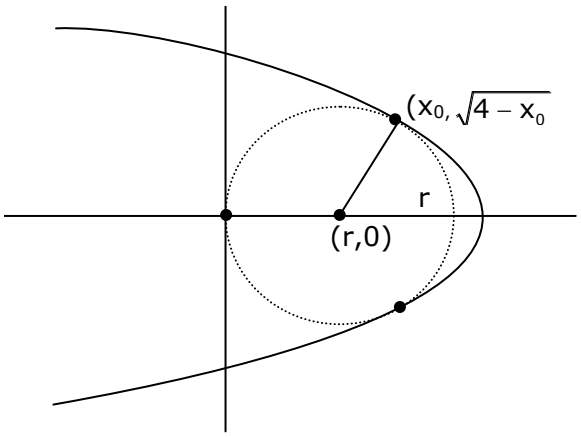
Ans. 1.5

522510 **Parabola, Ellipse & Hyperbola**

8. The value of α is ___ .
C का मान _____ है।

Ans. 2

Sol.



$(x-r)^2 + y^2 = r^2$

$x^2 + r^2 - 2xr + 4 - x = r^2$

$$\frac{x^2+4-x}{2x} = r \qquad \ldots.(1)$$

$$\therefore \frac{x_0^2+4-x_0}{2x_0} = x_0 - \frac{1}{2}$$

$$\frac{x_0^2+4-x_0}{x_0} = 2x_0 - 1$$

$$x_0^2 + 4 - x_0 = 2x_0^2 - x_0$$

$$x_0^2 = 4$$

$$x_0 = 2$$

$$\text{Also, } 2y\frac{dy}{dx} = -1$$

$$m_t = -\frac{1}{2y_0}$$

$$m_n = +2y_0$$

$$y - \sqrt{4-x_0} = 2\sqrt{4-x_0}\,(x - x_0)$$

(r,0)

$$-\sqrt{4-x_0} = 2\sqrt{4-x_0}\,(r - x_0)$$

$$r = x_0 - \frac{1}{2} \qquad \ldots..(2) \qquad \therefore r = \frac{3}{2}$$

Question Stem for Question Nos. 9 and 10

Question Stem

Let $f_1:(0,\infty)\to\mathbb{R}$ and $f_2:(0,\infty)\to\mathbb{R}$ be defined by

$f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j\,dt,\quad x>0$

and $f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450,\ x>0,$

where, for any positive integer n and real numbers $a_1,a_2,\ldots,an$, $\prod_{i=1}^{n}a_i$ denotes the product of $a_1,a_2,\ldots,an$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i=1,2$, in the interval $(0,\infty)$.

प्रश्न संख्या 9 तथा 10 के लिए अनुच्छेद

माना $f_1:(0,\infty)\to\mathbb{R}$ तथा $f_2:(0,\infty)\to\mathbb{R}$

$f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j\,dt,\quad x>0$ तथा $f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450,\ x>0$, द्वारा परिभाषित है, जहाँ किसी धनात्मक पूर्णांक n तथा वास्तविक संख्याओं $a_1, a_2, \ldots\ldots a_n$ के लिए $\prod_{i=1}^{n}a_i$ है, जो $a_1, a_2, \ldots\ldots a_n$ के गुणनफल को निरूपित करता है। माना m_i तथा n_i अंतराल $(0,\infty)$ में फलन f_i, $i=1,2$ के क्रमशः स्थानीय निम्निष्ठ बिंदुओं तथा स्थानीय उच्चिष्ठ बिंदुओं की संख्या को निरूपित करते हैं, तब

522581 **Definite Integration**

9. The value of $2m_1+3n_1+m_1n_1$ is ____ .

$2m_1+3n_1+m_1n_1$ का मान ____ है।

Ans. 57

522601 **Definite Integration**

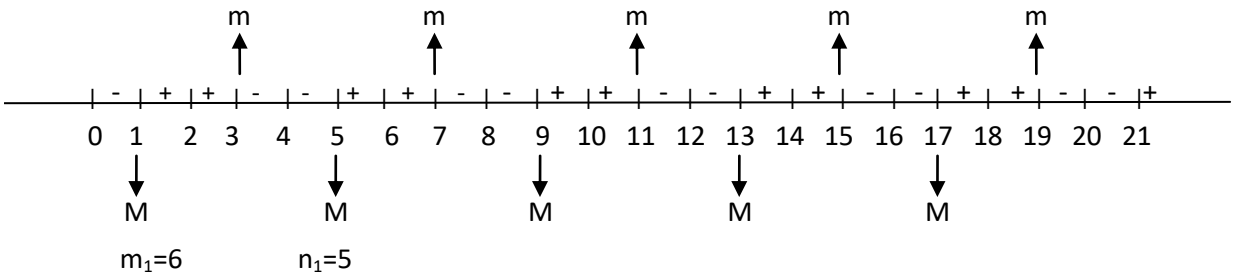
10. The value of $6m_2+4n_2+8m_2n_2$ is ____.

$6m_2+4n_2+8m_2n_2$ का मान ____ है।

Ans. 6

Sol. $f_1(x)=\int_0^x(t-1)(t-2)^2(t-3)^3\ldots\ldots(t-21)^{21}\,dt$

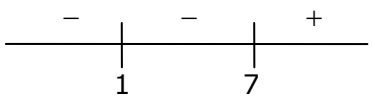
$\frac{df_1}{dx}=(x-1)(x-2)^2(x-3)^3\ldots\ldots(x-21)^{21}=0$



$$f_2'(x) = 98\times 50(x-1)^{49} - 600\times 49(x-1)^{48}$$

$$= (x-1)^{48}\{4900(x-y) - 29400\}$$

$$= (x-1)^{48}\{4900x - 34300\}$$



$m_2=1$
$n_2=0$

(9) 12+15+30 = 57.

(10) 6 + 0 = 6.

Question Stem for Question Nos. 11 and 12

Question Stem

Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}, i=1,2$, and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$ be functions such that

$g_1(x)=1$, $g_2(x)=|4x-\pi|$ and $f(x)=\sin^2 x$, for all $x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$

Define

$$S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x).g_i(x)\,dx, i=1,2$$

प्रश्न संख्या 11 तथा 12 के लिए अनुच्छेद

माना $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}, i=1,2$ तथा $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$

फलन है जो इस प्रकार है कि सभी $\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$ के लिए $g_1(x)=1$, $g_2=|4x-\pi|$ तथा $f(x)=\sin^2 x$ है।

$S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x).g_i(x)\,dx, i=1,2$ को परिभाषित कीजिए।

522609 **Definite Integration**

11. The value of $\frac{16S_1}{\pi}$ is _____.

$\frac{16S_1}{\pi}$ का मान _____ है।

Sol. **2**

$$S_1 = \int_{\pi/8}^{3\pi/8} \sin^2 x \, dx$$

$$\frac{1}{2}\int_{\pi/8}^{3\pi/8} (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \left(\frac{\sin 2x}{2}\right)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\right] = \frac{\pi}{8} - \frac{1}{2}\left[\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right]$$

$$= \frac{\pi}{8} \quad \Rightarrow \frac{16 \times \frac{\pi}{8}}{\pi} = 2$$

522612 **Definite Integration**

12. The value of $\frac{48S_2}{\pi^2}$ is _____.

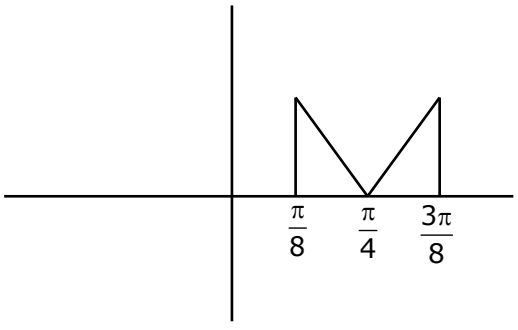
$\frac{48S_2}{\pi^2}$ का मान _____ है।

Ans. **1.5**

$$S_2 = \int_{\pi/8}^{3\pi/8} \sin^2 |4x-\pi|\,dx$$

$$S_2 = \int_{\pi/8}^{3\pi/8} \cos^2 x\,|2\pi-\pi-4x|\,dx$$

$$\int_{\pi/8}^{3\pi/8} \left(\sin^2 x\,|4x-\pi| + \cos^2 x\,|\pi-4x|\right)dx$$



$$2I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} |4x - \pi|$$

$$2I = 2 \times \left\{ \frac{1}{2} \times \frac{\pi}{8} \times \frac{\pi}{2} \right\}$$

$$2I = \frac{\pi^2}{16}$$

$$S_2 = \frac{\pi^2}{32}$$

$$\therefore \frac{48S_2}{\pi^2} = 1.5$$

Section – 3

- This section contains TWO (02) paragraphs. Based on each paragraph, there are TWO (02) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONLY ONE of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −1 In all other cases.

Paragraph

Let

$$M=\{(x,y)\in\mathbb{R}\times\mathbb{R}:x^2+y^2\leq r^2\},$$

where $r>0$. Consider the geometric progression $a_n=\frac{1}{2^{n-1}}$, $=1,2,3,\ldots$. Let $S_0=0$ and, for $n\geq1$, let S_n denote the sum of the first n terms of this progression. For $n\geq1$, let C_n denote the circle with center $(S_{n-1},0)$ and radius a_n, and D_n denote the circle with center (S_{n-1},S_{n-1}) and radius a_n.

माना $M = \{(x,y) \in R \times R : x^2 + y^2 \leq r^2\}$,

जहाँ $r > 0$ है। गुणोत्तर श्रेणी $a_n = \frac{1}{2^{n-1}}$, $n = 1,2,3,...$ पर विचार कीजिए। माना $S_0 = 0$ तथा $n \geq 1$के लिए, माना S_n इस श्रेणी के प्रथम n पदों के योगफल को निरूपित करता है। $n \geq 1$ के लिए माना C_n उस वृत्त को निरूपित करता है जिसका केन्द्र $(S_{n-1}, 0)$ है तथा त्रिज्या a_n है तथा D_n उस वृत्त को निरूपित करता है जिसका केन्द्र (S_{n-1}, S_{n-1}) तथा त्रिज्या a_n है।

522618 **Circle**

13. Consider M with $r=\frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A)$k+2l=22$ (B)$2k+l=26$ (C)$2k+3l=34$ (D)$3k+2l=40$

M पर विचार कीजिए जहाँ $r=\frac{1025}{513}$ है। माना k उन सभी वृत्तों C_n की संख्या है जो M के अंदर स्थित है। माना l इन k वृत्तों में से उन वृत्तों की अधिकतम संख्या है जिसमें कोई दो वृत्त प्रतिच्छेद नहीं करते हैं। तब

(A)$k+2l=22$ (B)$2k+l=26$ (C)$2k+3l=34$ (D)$3k+2l=40$

Ans. D

$$S_n = 2\left(1-\frac{1}{2^n}\right)$$

C_1 Centre (0,0), r = 1

C_2 Centre (1,0), r = $\frac{1}{2}$

C_3 Centre ($\frac{3}{2}$,0), r = $\frac{1}{4}$

$r = \frac{1025}{513}$

$x^2 + y^2 \le r^2$

$S_{n-1} + r_2 < r_1$

$S_{n-1} + a_n < \frac{1025}{513}$

$\Rightarrow S_n < \frac{1025}{513} \Rightarrow 1-\frac{1}{2^n} < \frac{1025}{1026}$

$\frac{1}{2^n} > \frac{1}{1026} \Rightarrow 2^n < 1026$

n = 10

$C_1, C_3, C_5, C_7, C_9 \rightarrow$ do not intersect

$C_2, C_4, C_6, C_8, C_{10} \rightarrow$ do not intersect

$\therefore$ maximum 5 circles do not intersect

l = 5

3k + 2l = 40

522628 **Circle**

14. Consider M with $r=\frac{\left(2^{199}-1\right)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199 (C) 200 (D) 201

M पर विचार कीजिए, जहाँ $r=\frac{\left(2^{199}-1\right)\sqrt{2}}{2^{198}}$ तब उन सभी वृत्तों D_n की संख्या कौनसी है जो M के अंदर स्थित है?

(A) 198 (B) 199 (C) 200 (D) 201

Ans. B

n = 1 $(S_0, S_0) = (0,0)$ centre

r = 1

n = 2 (S_1, S_1) = centre (1,1)

$r_2 = \frac{1}{2}$

$$\therefore \sqrt{2}S_{n-1} + a_n < \frac{\left(2^{199}-1\right)\sqrt{2}}{2^{198}}$$

$$2\sqrt{2}\left(1-\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}} < \frac{\left(2^{199}-1\right)\sqrt{2}}{2^{198}}$$

$$2\sqrt{2}-\frac{\sqrt{2}}{2^{n-2}}+\frac{1}{2^{n-1}} < 2\sqrt{2}-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}}\left(\frac{1}{2}-\sqrt{2}\right) < \frac{-\sqrt{2}}{2^{198}}$$

$$2^{n-2} < \left(2-\frac{1}{\sqrt{2}}\right)^2$$

$\therefore$ n ≤ 199

Paragraph

Let $\psi_1:[0,\infty)\to\mathbb{R}$, $\psi_2:[0,\infty)\to\mathbb{R}$, $f:[0,\infty)\to\mathbb{R}$ and $g:[0,\infty)\to\mathbb{R}$ be functions such that $f(0)=g(0)=0$,

$$\psi_1(x)=e^{-x}+x,\ x\geq 0,$$

$$\psi_2(x)=x^2-2x-2e^{-x}+2,\ x\geq 0,$$

$$(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}\,dt,\ x>0$$

and $$g(x)=\int_0^{x^2}\sqrt{t}\ e^{-t}\,dt,\ x>0.$$

अनुच्छेद

माना $\psi_1:[0,\infty)\to\mathbb{R}$, $\psi_2:[0,\infty)\to\mathbb{R}$, $f:[0,\infty)\to\mathbb{R}$ तथा $g:[0,\infty)\to\mathbb{R}$ फलन है जो इस प्रकार है कि f(0)=g(0)=0,

$$\psi_1(x)=e^{-x}+x,\ x\geq 0,$$

$$\psi_2(x)=x^2-2x-2e^{-x}+2,\ x\geq 0,$$

$$(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}\,dt,\ x>0$$

तथा $g(x)=\int_0^{x^2}\sqrt{t}\ e^{-t}\,dt,\ x>0.$

522654 **Definite Integration**

15. Which of the following statements is TRUE ?

(A) $f\left(\sqrt{\ln 3}\right)+g\left(\sqrt{\ln 3}\right)=\frac{1}{3}$

(B)For every $x>1$, there exists an $\alpha\in(1,x)$ such that $\psi_1(x)=1+\alpha x$

(C)For every $x>0$, there exists a $\beta\in(0,x)$ such that $\psi_2(x)=2x(\psi_1(\beta)-1)$

(D)f is an increasing function on the interval $\left[0,\frac{3}{2}\right]$.

(A) $f\left(\sqrt{\ln 3}\right)+g\left(\sqrt{\ln 3}\right)=\frac{1}{3}$

(B) प्रत्येक $x>1$ के लिए, एक $\alpha\in(1,x)$ इस प्रकार विद्यमान है कि $\psi_1(x)=1+\alpha x$

(C) प्रत्येक $x>0$ के लिए एक $\beta\in(0,x)$ इस प्रकार विद्यमान है कि $\psi_2(x)=2x(\psi_1(\beta)-1$

(D) अंतराल $\left[0,\frac{3}{2}\right]$ में f एक वर्धमान फलन है

Ans. C

(A) $f(x)=2\int_0^x\left(t-t^2\right).e^{-t^2}dt$

$g(x)=\int_0^{x^2}\sqrt{t}\cdot e^{-t}dt$ $\quad t=u^2$

$dt=2udu$

$g(x)=2\int_0^x u^2\cdot e^{-u^2}du$

$f(x)+g(x)=2\int_0^x\left(t-t^2+t^2\right)e^{-t^2}dt$

$=2\int_0^x t\cdot e^{-t^2}dt$ $\quad t^2=z$

$2tdt=dz$

$=2\int_0^{x^2}e^{-z}dz$ $\quad=-\left(e^{-z}\right)_0^{x^2}$

$=-\left\{e^{-x^2}-1\right\}$

$f(x)+g(x)=1-e^{-x^2}$

$\therefore f\left(\sqrt{\ln 3}\right)+g\left(\sqrt{\ln 3}\right)=1-e^{-\ln 3}=1-\frac{1}{3}=\frac{2}{3}$

(B) $e^{-x}+x=1+\alpha x$

$e^{-x}-1=(\alpha-1)x$

LHS < 0 and RHS > 0 (not possible)

(C) $\psi_2'(x) = 2x - 2 + 2e^{-x}$

$= 2\psi_1(x) - 2$ (by LMVT)

$\psi_2'(\beta) = \dfrac{\psi_2(x) - \psi_2(0)}{x - 0}$

$2\psi_1(\beta) - 2 = \dfrac{x^2 - 2x - 2e^{-x} + 2}{x}$

$2x(\psi_1(\beta) - 1) = \psi_2(x)$

Hence its true

(D) $f(x) = 2\int_0^x (t - t^2)\cdot e^{-t^2}$

$f'(x) = 2(x - x^2).e^{-x^2}$

$= 2x(1 - x)e^{-}$

(+ve) | (+ve)

Changes sign

522664 **Definite Integration**

16. Which of the following statements is TRUE ?

(A)$\psi_1(x) \le 1$, for all $x>0$ (B)$\psi_2(x) \le 0$, for all $x>0$

(C)$(x) \ge 1 - e^{-x^2} - \dfrac{2}{3}x^3 + \dfrac{2}{5}x^5$, for all $x \in \left(0, \dfrac{1}{2}\right)$ (D) $g(x) \le \dfrac{2}{3}x^3 - \dfrac{2}{5}x^5 + \dfrac{1}{7}x^7$, for all $x \in \left(0, \dfrac{1}{2}\right)$

निम्न में से कौनसा कथन सत्य है?

(A)$\psi_1(x) \le 1$, सभी $x>0$ के लिए (B) $\psi_2(x) \le 0$ सभी $x>0$ के लिए

(C)$(x) \ge 1 - e^{-x^2} - \dfrac{2}{3}x^3 + \dfrac{2}{5}x^5$ सभी $x \in \left(0, \dfrac{1}{2}\right)$ के लिए (D) $g(x) \le \dfrac{2}{3}x^3 - \dfrac{2}{5}x^5 + \dfrac{1}{7}x^7$, सभी $x \in \left(0, \dfrac{1}{2}\right)$ के लिए

Ans. D

(A) $e^{-x} + x - 1 \le 0 \quad \forall\, x>0$

$h(x) = e^{-x} + x - 1$

$h'(x) = -e^{-x} + 1$

$= 1 - \dfrac{1}{e^x} = \dfrac{e^x - 1}{e^x}$

$\Rightarrow$ false

(B) $x^2 - 2x - 2e^{-x} + 2 \le 0 \qquad \forall\, x>0$

False as $x \to \infty$

LHS $\to \infty$

(C) $f(x) + g(x) = 1 - e^{-x^2}$

$f(x) = 1 - e^{-x^2} - g(x)$ ← +ve function

$\therefore \left(\frac{2}{3}x^3 - \frac{2}{5}x^5\right) \to +ve, \qquad x \in \left(0, \frac{1}{2}\right)$

$f(x) \le 1 - e^{-x^2} - \left(\frac{2x^3}{3} - \frac{2}{5}x^5\right)$

$g(x) \ge \int_0^{x^2} \sqrt{t}(1-t)$

$\ge \int_0^{x^2} \left(\sqrt{t} - t^{3/2}\right) dt$

$\ge \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{5}{2}}}{\frac{5}{2}}\right) x^2$

$g(x) \ge \left(\frac{2}{3}x^3 - \frac{2}{5}x^5\right)$

$\Rightarrow$ False

(D) $e^{-t} \le \left(1 - t + \frac{t^2}{2}\right)$

$\int_0^{x^2} \sqrt{t} e^{-t} \le \int^{x^2} \sqrt{t} \cdot \left(1 - t + \frac{t^2}{2}\right) dt$

$g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$

$\Rightarrow$ true

SECTION 4

- This section contains THREE (03) questions.
- The answer to each question is a NON-NEGATIVE INTEGER.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If ONLY the correct integer is entered;

  Zero Marks : 0 In all other cases.

522524 **Probability**

17. A number is chosen at random from the set {1,2,3,…,2000}. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of 500p is ___ .

समुच्चय {1,2,3,…,2000} में से एक संख्या का यादृच्छया चयन किया जाता है। माना चयनित संख्या के 3 या 7 का गुणज होने की प्रायिकता p है। तब 500p का मान ____ है।

Ans. 214

$$\left[\frac{2000}{3}\right]+\left[\frac{2000}{7}\right]-\left[\frac{2000}{21}\right]$$

666 + 285 −95

$\therefore$ Prob. = $\frac{856}{2000}$

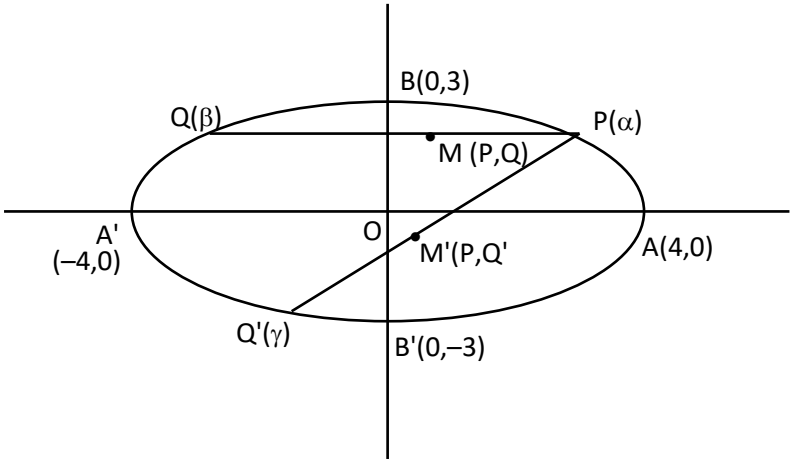
500P = 214

522530 **Parabola, Ellipse & Hyperbola**

18. Let E be the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. For any three distinct points P,Q and Q' on E, let M(P, Q) be the mid-point of the line segment joining P and Q, and M(P, Q') be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between M(P, Q) and M(P, Q'), as P, Q and Q' vary on E, is ___.

माना E एक दीर्घवृत्त $\frac{x^2}{16}+\frac{y^2}{9}=1$ है। E पर किन्हीं तीन भिन्न बिन्दुओं P,Q तथा Q' के लिए, माना P एवं Q को जोड़ने वाले रेखाखण्ड का मध्य बिन्दु M(P, Q) है तथा P एवं Q' को जोड़ने वाले रेखाखण्ड का मध्य बिन्दु M(P, Q') है। तब M(P, Q) तथा M(P, Q') के मध्य अधिकतम संभव दूरी का मान है, (E पर P,Q एवं Q' घुमते रहते हैं).

Ans. 4



$$M\left(\frac{4a\alpha+4a\beta}{2},\frac{4\sin\alpha+3\sin\beta}{2}\right)$$

$$M'\left(\frac{4a\alpha+4a\gamma}{2},\frac{4\sin\alpha+3\sin\gamma}{2}\right)$$

$$MM'=\frac{1}{2}\sqrt{(4\cos\beta-4\cos\gamma)^2+(3\sin\beta-3\sin\gamma)^2}$$

$$MM'=\frac{1}{2}\text{distance between Q \& Q'}$$

MM' is free from 'P'

Max of 'QQ' will be possible if QQ' is major axis

$\Rightarrow$ $QQ'=2\times4=8$

$\Rightarrow$ $MM'=\frac{1}{2}\times8=4$

522537 **Definite Integration**

19. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If

$$I=\int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx,$$

then the value of 9I is _____.

किसी वास्तविक संख्या x के लिए, माना [x], x के बराबर या इससे कम महत्तम पूर्णांक को निरूपित करता है। यदि $I=\int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, तब 9I का मान _____ है।

Ans. **182**

1. $\frac{10x}{x+1}=1$

$10x = x+1$

$$x = \frac{1}{9}$$

2. $\frac{10x}{x+1} = 4$

$10x = 4x + 4$

$6x = 4$

$$x = \frac{2}{3}$$

3. $\frac{10x}{x+1} = 9$ $\quad 10x = 9x + 9$

$x = 9$

$$\therefore \int_0^{\frac{1}{9}} 0dx + \int_{\frac{1}{9}}^{\frac{2}{3}} 1dx + \int_{\frac{2}{3}}^{9} 2dx + \int_{9}^{10} 3dx$$

$$\left(\frac{2}{3} - \frac{1}{9}\right) + 2\left(9 - \frac{2}{3}\right) + 3$$

$$\frac{6-1}{9} + 2\left(\frac{25}{3}\right) + 3$$

$$\frac{5}{9} + \frac{50}{3} + 3$$

$$\left(\frac{5+150+27}{9}\right) \cdot 9$$

$=182$