NMTC_2017

(NATIONAL MATHEMATICS TALENT CONTEST)

JUNIOR LEVEL - IX & X STANDARDS

PART - A

1. If m is a real number such that $m^2 + 1 = 3m$, the value of $\frac{2m^5 - 5m^4 + 2m^3 - 8m^2}{m^2 + 1}$ is:

(A) 1 (B) 2 (C) –1 (D) –2

Sol. $m^2 + 1 = 3\,m$

$$\frac{2m^5 - 5m^4 + 2m^3 - 8m^2}{m^2 + 1}$$

$$\begin{array}{r|l}
m^2+1 & 2m^5 - 5m^4 + 2m^3 - 8m^2 \\
 & 2m^5 \qquad\quad + 2m^2 \\
 & \underline{- \qquad\qquad -} \\
 & \qquad -5m^4 - 8m^2 \\
 & \qquad -5m^4 - 5m^2 \\
 & \qquad \underline{+ \qquad\quad +} \\
 & \qquad\qquad -3m^2
\end{array}$$

$2m^3 - 5m^2 - \frac{3m^2}{m^2 + 1}$

$2m^3 - 5m^2 - \frac{3m^2}{3m}$

$2m^2 - 5m^2 - m$

$m\,(2m^2 - 5m - 1)$

$m\,(2(3m-1) - 5m - 1)$

$m\,(6m - 2 - 5m - 1)$

$m(m-3)$

$m^2 - 3m = -1$

2. Consider the equation $\frac{7x}{2} - a = \frac{5x}{3} + 9$. The least positive a for which the solution x to the equation is a positive integer is:

(A) 1 (B) 2 (C) 3 (D) 4

Sol. $\frac{7x}{2} - a = \frac{5x}{3} + 9$

$\frac{7x}{2} - \frac{5x}{3} + a + 9$

$\frac{21x - 10x}{6} = a + 9$

$\frac{11x}{6} = a + 9$

$11x = 6 \times (a + 9)$

a = 2

satisfy the equation

3. If x = 2017 and y = $\frac{1}{2017}$, the value of $\left\{\frac{\frac{x}{y}+2}{\frac{x}{y}+1} + \frac{x}{y}\right\} \div \left\{\frac{x}{y} + 2 - \frac{\frac{x}{y}}{\frac{x}{y}+1}\right\}$ is:

(A) 2017 (B) 2017^2 (C) $\frac{1}{2017^2}$ (D) 1

Sol. x = 2017 and y = $\frac{1}{2017}$

$$\left(\frac{\frac{x}{y}+2}{\frac{x}{y}+1} + \frac{x}{y}\right) \div \left[\frac{x}{y} + 2 - \frac{\frac{x}{y}}{\frac{x}{y}+1}\right] \div \left[\frac{x}{y} + \frac{\frac{2x}{y} + 2 - \frac{x}{y}}{\frac{x}{y}+1}\right]$$

$$\frac{\left(\frac{\frac{x}{y}+2}{\frac{x}{y}+1} + \frac{x}{y}\right)}{\left(\frac{x}{y} + \frac{x/y + 2}{\frac{x}{y}+1}\right)} = 1$$

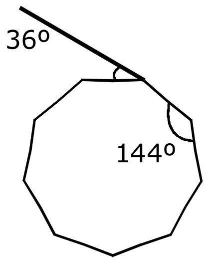
4. The ratio of an interior angle of a regular pentagon to an exterior angle of a regular decagon is:

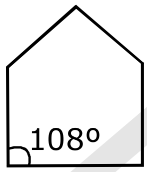
(A) 4 : 1 (B) 3 : 1 (C) 2 : 1 (D) 7 : 3

Sol. Inferior angle of polygon

$\frac{180(n-2)}{n}$ n = no of sides

exterior angle = $\frac{360}{n}$





Sum of all interior angle of pentagon is 540º

108 :36

3 : 1

5. The smallest integer x which satisfies the inequality $\frac{x-5}{x^2+5x-14} > 0$ is:

(A) –8 (B) –6 (C) 0 (D) 1

Sol. $\frac{x-5}{x^2-5x-14} > 0$

$\frac{x-5}{x^2-7x-2x-14} > 0$

$\frac{x-5}{x(x+7)-2(x+7)} > 0$

$\frac{x-5}{(x+2)(x+7)} > 0$

x = –6 satisfy the given inequality

6. If x and y satisfy the equations:

$\sqrt{\frac{20y}{x}} = \sqrt{x+y} + \sqrt{x-y}$, $\sqrt{\frac{16x}{5y}} = \sqrt{x+y} - \sqrt{x-y}$ the value of $x^2 + y^2$ is:

(A) 2 (B) 16 (C) 25 (D) 41

Sol. $\sqrt{\frac{20y}{x}} = \sqrt{x+y} + \sqrt{x-y}$ ....(i) Find x^2+y^2

$\sqrt{\frac{16x}{5y}} = \sqrt{x+y} - \sqrt{x-y}$ ....(ii)

(i) × (ii)

$(x+y)-(x-y) = \sqrt{\frac{20y}{x} \times \frac{16x}{5y}}$

2y = 8

y = 4

(i) + (ii)

$2\sqrt{x+y} = \sqrt{\frac{20y}{x} \times \frac{16x}{5y}}$

both side square

$4(x+y) = \frac{20y}{x} + \frac{16x}{5y} + 2\sqrt{\frac{20y}{x} \times \frac{16x}{5y}}$

put y = 4

$4(x+4) = \frac{20 \times 4}{x} + \frac{16x}{20} + 2 \times 8$

$(x+4) = \frac{20}{x} + \frac{x}{5} + 4$

$5x(x+4) = 100 + x^2 + 20x$

$5x^2 + 20x = 100 + x^2 + 20x$

$4x^2 = 100$

$x^2 = 25$

Value of $x^2 + y^2 = 25+16 = 41$

7. 125% of a number x is y. What percentage of 8y is 5x?

(A) 30% (B) 40% (C) 50% (D) 60%

Sol. $\frac{125}{100} \times x = y$

$\frac{5x}{4} = y$

5x = 4y

Let required % is a than

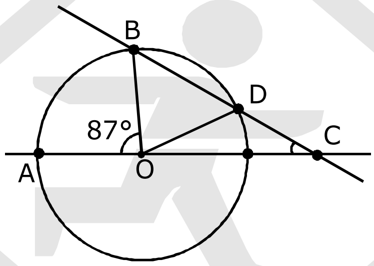
$8y \times \frac{a}{100} = 5x$

Put 8y = 10x

$10x \times \frac{9}{100} = 5x$

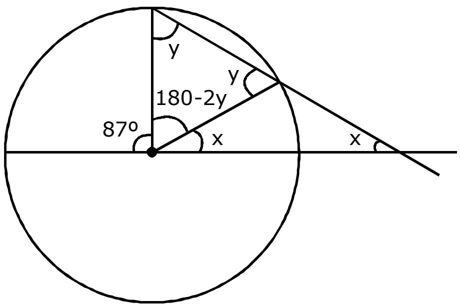
a = 50%

8. In the adjoining figure, O is the center of the circle adn OD = DC. If ∠AOB = 87°, the measure of the angle ∠OCD is:



(A) 27° (B) 28° (C) 29° (D) 19°

Sol.



from the exterior angle property y = 2x

87° + 180 – 2y + x = 180

87° = 2y–x

87° = 2(2x) – x

87° = 4x–x

87° = 3x

x = 29°

9. a, b, c, d, e are real numbers such that $\frac{a}{b}=\frac{2}{3},\frac{b}{c}-\frac{1}{3},\frac{c}{d}=\frac{1}{4},e=\frac{ac}{b^2+c^2}$. The value of e is:

(A) $\frac{1}{9}$ (B) $\frac{2}{9}$ (C) $\frac{1}{5}$ (D) $\frac{2}{5}$

Sol. $\frac{a}{b}=\frac{2}{3},\ \frac{b}{c}=\frac{1}{3},\ \frac{c}{d}=\frac{1}{4}$

$e = \frac{ac}{b^2+c^2}$ ....(A)

$\frac{a}{b}=\frac{2}{3}$ ....(i) $\frac{b}{c}=\frac{1}{3}$ ....(ii) $\frac{c}{d}=\frac{1}{4}$ ....(iii)

multiply (i) × (ii)

$\frac{a}{c}=\frac{2}{3}\times\frac{1}{3}$

$\frac{a}{c} = \frac{2}{9}$

$a = \frac{2c}{9}$ ....(iv)

Put in eq. no. (v)

$\frac{\frac{a}{b}}{\frac{b}{c}}=\frac{\frac{2}{3}}{\frac{1}{3}}$ $\frac{a^c}{b^2}=2$

$ac = 2b^2$ ....(v)

$\frac{2c}{9}\times c = 2b^2$

$\frac{2c^2}{9a}=2b^2$ $c^2 = 9b^2$

$e = \frac{ac}{b^2+9b^2}$

$e = \frac{ac}{10b^2}$

$e = \frac{2b^2}{10b^2}$

$e = \frac{1}{5}$

10. The length and breadth of a rectangular field are integers. The area is numerically 9 more than the perimeter. The perimeter is:

(A) 24 (B) 32 (C) 36 (D) 40

Sol. If length = 15

Breadth = 3

According to question

$15 \times 3 = 2(15 + 3) + 9$

$45 = 2(18) + 9$

$45 = 36 + 9$

$45 = 45$

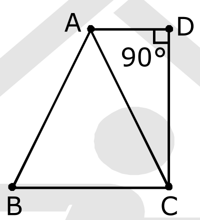
Perimeter

$= 2(\ell + b)$

$= 2(18)$

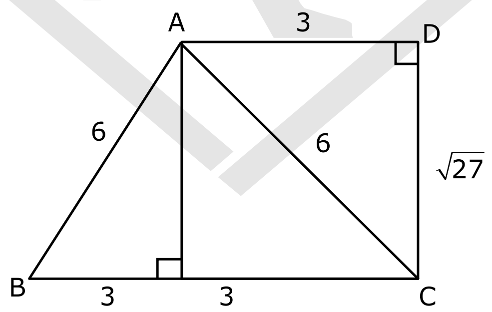
$= 36$

11. ABCD is a trapezium in which ABC is an equilateral triangle with area $9\sqrt{3}$ square units. If $\angle ADC = 90°$, the area of the trapezium in square units is:



(A) $12\sqrt{3}$ (B) $\frac{15\sqrt{3}}{2}$ (C) $\frac{27\sqrt{3}}{2}$ (D) $\frac{35\sqrt{3}}{2}$

Sol. area $a\sqrt{3}$



area = $\frac{\sqrt{3}}{4}a^2 = a\sqrt{3}$

$a^2 = 36$

$a = 6$

area of triangle ACD

$= \frac{1}{2} \times 3 \times \sqrt{27}$

$= \frac{3}{2} \times 3\sqrt{3}$

Area of trapezium

Area of ΔABC + Area of ΔACD

$9\sqrt{3} + \frac{9}{2}\sqrt{3}$

$= \frac{27}{2}\sqrt{3}$

12. p is a prime number such that $p^2 - 8p - 65 > 0$. The smallest value of p is:

(A) 7 (B) 11 (C) 13 (D) 17

Sol. $p^2 - 8p - 65 > 0$

$p^2 - 13p + 8p - 65 > 0$

$p(p-13) + 8(p-13) > 0$

$(p+8)(p-13) > 0$

$p > -8$

$p > 13$ from the above p = 17 satisfy the condition

13. The least positive integer n such that $2015^n + 2016^n + 2017^n$ is divisible by 10 is:

(A) 1 (B) 3 (C) 4 (D) None of these

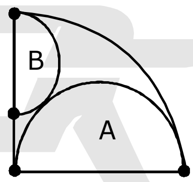
Sol. $2015^n + 2016^n + 2017^n$ is divisible by 10

For divisible by 10

Last digit required to be = 0

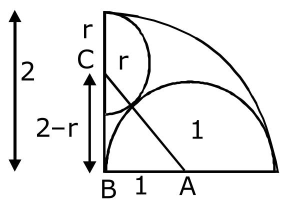
that cannot satisfy by the all three options so ans is "none of these"

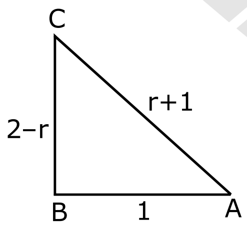
14. n a quadrant of a circle of diameter 4 units semicircles are drawn as shown. The radius of the smallest circle (B) is:



(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Sol.





$(r + 1)^2 = 1^2 + (2 - r)^2$

$r = 2/3$

15. The product of two positive integers is twice their sum; the product is also equal to six times the difference between the two integers. The sum of these integers is:

(A) 6 (B) 7 (C) 8 (D) 9

Sol. $xy = 2(x+y)$ ....(1)

$xy = 6(x+y)$

$2x+2y = xy$ ....(i)

$6x-6y = xy$ ....(ii)

(ii) – (i)

$4x-8y = 0$

$x-2y = 0$

$x = 2y$

from (i)

$2y\times y = 2\,(2y+y)$

$2y^2 = 6y$

$2y^2-6y = 0$

$y^2-3y = 0$

$y(y-3) = 0$

$y = 0$

$y = 3$

$3x = 2\,(x+3)$

$3x = 2x + 6$

$x = 6$

$x + y = 9$

PART - B

16. n is a natural number such that n minus 12 is the square of an integer and n plus 19 is the square of another integer. The value of n is ________.

Sol. If n = 237

n-12 = 237 – 12

= 225

$= 15^2$

n + 99 = 237 + 19

= 256

$= 16^2$

17. The number of three digit numbers which have odd number of factors is _______.

Sol. Odd factors

(a) If there are odd factor of on integer that integer will be square

(b) If integer has 3 factors only. If is square of a prime number

From 100 for 999 the perfect squares are

31 – 10 + 1 = 22 numbers

So 22 odd no. of factors

18. The positive integers a, b, c are connected by the inequality $a^2 + b^2 + c^2 + 3 < ab + 3b + 2c$, then the value of a + b + c is _______.

Sol. 4

19. The sum of all roots of the equation $|3x - |1 - 2x|| = 2$ is _______.

Sol. $|\ 3x - |\ |1-2x|\ |\ = 2$

We know that

$|x| = -x$ $\{x<0,\ x>0\}$

$|x| = x$

(i) $3x-1+2x = 2$

$5x - 1 = 2$

$5x = 3$

$x = \frac{3}{5}$

(ii) $3x+1-2x = 2$

$x+1 = 2$

$x = 1$

(iii) $-\{3x-(1-2x)\} = 2$

$3x-1+2x = -2$

$5x-1 = -2$

$5x = -1$

$x = -\frac{1}{5}$

(iv) $-\{3x+(1-2x)\} = 2$

$3x+1-2x = -2$

$x+1 = -2$

$x = -3$

Sum of roots

$\frac{3}{5} + 1 - \frac{1}{5} - 3$

$\frac{3+5-1-15}{5} = \frac{8-16}{5} = \frac{-8}{5}$

20. PQR is a triangle with PQ = 15, QR = 25, RP = 30. A, B are points on PQ and PR respectively such that $\angle PBA = \angle PQR$. The perimeter of the triangle PAB is 28, then the length of AB is _______.

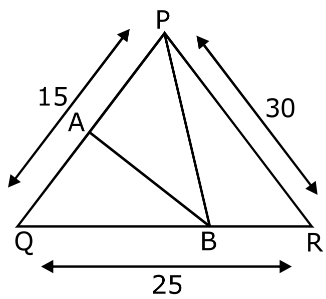
Sol. In ΔPBA & PQR

$\angle A = \angle P$

$\angle B = \angle Q$

By AA criteria similarity

$\Delta PBA \sim \Delta PQR$



$\frac{PB}{PQ} = \frac{BA}{QR} = \frac{PA}{PR}$

$\frac{PB}{15} = \frac{BA}{25} = \frac{PA}{30} = K$

PB = 15K BA = 25K

PA = 30K

Also we know that PB + BA + PA = 28

15K + 25K + 30K = 28

70K = 28

$K = \frac{28}{70}$

$AB = 25K = 25 \times \frac{28}{70}$

AB = 10

21. A hare sees a hound 100 m away from her and runs off in the opposite direction at a speed of 12 KM an hour. A minute later the hound perceives her and gives a chase at a speed of 16 KM an hour. The distance at which the hound catches the hare (in meters) is _______.

Sol. 12 km/hr

$= 12 = \frac{1000}{60 \times 60}$

$= 12 \times \frac{5}{18}$

$= \frac{10}{3} \text{ m/sec}$

16 km/hr

$= 16 \times \frac{5}{18}$

$= \frac{40}{9} \text{ m/sec}$

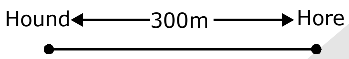
Distance travel by hore in 1 minute = $60 \times \frac{10}{3} = 200$ meter

So distance between hound and hore is 100 + 200 = 300 meters

Relative velocity of hore and hound is (16 – 12) km/hr = 4 km/hr

$= 4 \times \frac{5}{18}$

$= \frac{10}{9} \text{ m/sec}$



Time taken to reach hound hear to hore = $\frac{300}{10/9} = 270$ sec

Distance travelled the hound to catch the hore is = $\frac{40}{9} \times 270$ = 1200 meters

22. Two circles touch both the arms of an angle whose measure is 60°. Both the circles also touch each other externally. The radius of the smaller circle is r. The radius of the bigger circle (in term of r) is _______.

Sol. 3r

23. a, b are distinct natural numbers such that $\frac{1}{a} + \frac{1}{b} = \frac{2}{5}$. If $\sqrt{a+b} = k\sqrt{2}$, the value of k is ______.

Sol. $\frac{1}{a} + \frac{1}{b} = \frac{2}{5}$ $\qquad \sqrt{a+b} = K\sqrt{2}$

$\frac{a+b}{ab} = \frac{2}{5}$ $\qquad \left(\sqrt{a+b}\right)^2 = 2K^2$

a + b = $\frac{2}{5}$ab

$\frac{2}{5}ab = 2K^2$

$K^2 = \frac{ab}{5}$

$\Rightarrow$ K^2 must be a proper square

$\Rightarrow$ $K^2 = 1, 4, 9, 16$ _______ so on

$\Rightarrow$ $a = 1$ and $b = 5$

But it will not satify above condition

We can check for other values

Now let $K^2 = a$ $\Rightarrow$ $ab = 45$

a	b
1,	45
3,	15

$a = 3, b = 15$

$$\frac{1}{a}+\frac{1}{b}=\frac{2}{5}$$

$$\frac{1}{3}+\frac{1}{15}=\frac{2}{5}$$

$$\frac{15+3}{45}=\frac{2}{5}$$

$$\frac{18}{45}=\frac{2}{5}$$

$$\frac{2}{5}=\frac{2}{5}$$

$$\sqrt{a+b}=K\sqrt{2}$$

$$\sqrt{3+15}=K\sqrt{2}$$

$$\sqrt{18}=K\sqrt{2}$$

$$3\sqrt{2}=K\sqrt{2}$$

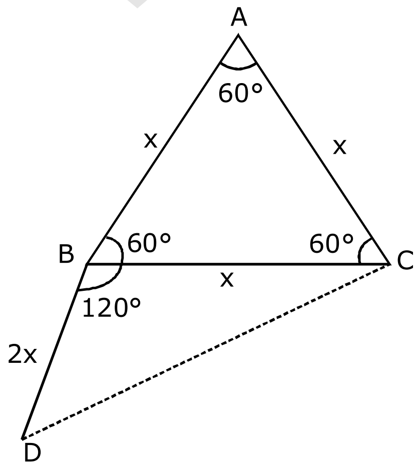
$K = 3$

24. The side AB of an equilateral triangle ABC is produced to D such that BD = 2AC. The value of $\frac{CD^2}{AB^2}$ is _________.

Sol. Now is ΔBDC

Applying cos formula

$$\cos 120° = \frac{BD^2+BC^2-CD^2}{2BD\times BC}$$



$$-\frac{1}{2} = \frac{(2x)^2+x^2-CD^2}{2(2x)(x)}$$

$$-\frac{1}{2} = \frac{4x^2+x^2-CD^2}{4x^2}$$

$-4x^2 = 2(4x^2 - CD^2)$

$-2x^2 = 5x^2 - CD^2$

$-7x^2 = -CD^2$

$CD^2 = 7x^2$

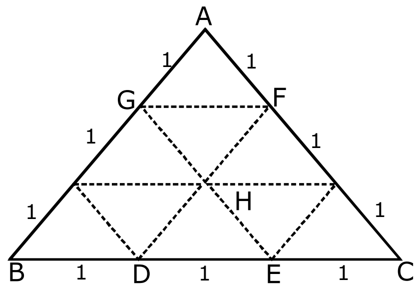
$$\frac{CD^2}{AB^2} = \frac{7x^2}{x^2}$$

$$\frac{CD^2}{AB^2} = \frac{7}{1}$$

25. D and E trisect the side BC of a triangle ABC. DF is drawn parallel to AB meeting AC at F. EG is drawn parallel to AC meeting AB at G. DF and EG cut at H. Then the numerical value of $\frac{\text{Area(ABC)}}{\text{Area(DHE)} + \text{Area(AFHG)}}$ is ________.

Sol. Area of ΔABC = 9 Area of ΔDHE

Area of ||gm AGHF = 2Area of ΔDHE



Now $\frac{\text{Area of } \Delta\text{ABC}}{\text{Area of } \Delta\text{DHE} + \text{Area of} || \text{gm AGHE}}$

$= \frac{9 \text{ Area of } \Delta\text{DHE}}{\text{Area of } \Delta\text{DHE} + 2 \text{ Area of } \Delta\text{DHE}}$

$= \frac{9 \text{ Area of } \Delta\text{DHE}}{3 \text{ Area of } \Delta\text{DHE}}$

$= \frac{3}{1}$

26. In an examination 70% of the candidates passed in English, 65% passed in Mathematics, 27% failed in both the subjects and 248 passed in both the subjects. The total number of candidates is _______.

Sol. Let the total no. of candidates are 'n'

The total no. of candidates passed in

English (E) = 0.7 n

no. of candidate passed in maths (m) = 0.65 n

no. of candidates neither passed in english nor in english $\bar{E} \cap \bar{m}$ = 0.27 n

no. of candiates passed passed both (E $\cap$ m) = 248

we know (E $\cap$ m) = n – ($\bar{E} \cap \bar{m}$)

and E $\cap$ m = (E) + (m) – (E $\cap$ m)

n–0.27n = 0.7n + 0.65n – 248

n = 400

2nd Solution

pass in English = 70%

fail = 30%

pass maths = 65

fail = 35%

fail in both = 27%

total no of fail student = fail in E + fail in M common

= 30 + 35 – 27

= 38%

It 38% fail than 62% will pass

Let total no. of student = x

hence

62% of (total student) = 248

$62 \times \frac{x}{100}$ = 248

x = 400

27. In a potato race, a bucket is placed at the starting point, which is 7m from the first potato. The other potatoes are placed 4m a part in a straight line from the first one. There are n potatoes in the line. Each competitor starts from the bucket, picks up the nearest potato, runs back with it, drops in the bucket, runs back to pick up the next potato, runs to the bucket and drops it and this process continues till all the potatoes are picket up and dropped in the bucket. Each competitor ran a total of 150m. The number of potatoes is _________.

Sol. For picking first potato, distance travelled = 14 m
As next potato is 4m away from previous one, total distance travelled is = 8m
1st round = 8
2nd round = 22
3rd round = 30
4th round = 38
5th round = 46
Total = 150

Or

It following pattern of an AP.
With a = 14, d = 8

Sum = $\frac{n}{2}[2a+(n-1)d]$

$150 = \frac{n}{2}[2\times 14+(n-1)8]$

Solving above equation, n = 5

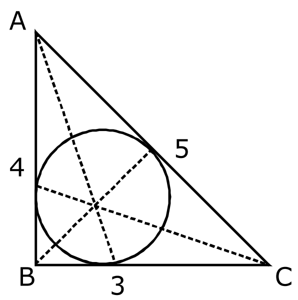
28. A two digit number is obtained by either multiplying the sum of its digits by 8 and adding 1, or by multiplying the difference of its digits by 13 and adding 2. The number is _________.

Sol. $8(x + y) + 1 \Rightarrow 8x + 8y + 1$
$13(x - y) + 2 \Rightarrow 13x - 13y + 2$
both are equal
$8x + 8y + 1 = 13x - 13y + 2$
$5x - 21y + 1 = 0$
From observation
x = 4 and y = 1
$5 \times 4 - 21 \times 1 + 1 = 0$
$20 - 21 + 1 = 0$
$21 - 21 = 0$
$0 = 0$
So, x = 4 and y = 1
No. is 41

29. The inradius of a right angled triangle whose legs have lengths 3 and 4 is _________.

Sol. Area of Δ = S.r ....(i)

$S = \frac{3+4+5}{2} = \frac{12}{2} = 6$



Area of Δ $= \sqrt{6(6-3)(6-4)(6-5)}$

$= \sqrt{6\times 3\times 2\times 1}$

$= \sqrt{36}$

$= 6$

6 = 6.r
r = 1

Or

$$\frac{1}{2}\times 3\times 4=6$$

$$=\frac{1}{2}\times r\times 3+\frac{1}{2}\times r\times 4+\frac{1}{2}\times r\times 5$$

6r = 6

r = 1 cm

30. a, b are positive reals such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$. If $\left(\frac{a}{b}\right)^3+\left(\frac{b}{a}\right)^3=2\sqrt{n}$, where n is a natural number, the value of n is ________.

Sol. $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$

$$\frac{b+a}{ab}=\frac{1}{a+b}$$

$(a+b)^2 = 2ab$

$a^2 + 2ab + b^2 = ab$

$a^2 + ab + b^2 = 0$

$\boxed{a^2+b^2=-ab}$ ....(i)

taking square both side

$a^4 + b^4 + 2a^2b^2 = a^2b^2$

$a^4+b^4+a^2b^2 = 0$

$a^4 + b^4 = -a^2b^2$ ....(ii)

put in eq. (i)

From the given

If $\left(\frac{a}{b}\right)^3+\left(\frac{b}{a}\right)^3=2\sqrt{n}$ n is natural no.

$$\left(\frac{a}{b}+\frac{b}{a}\right)\left[\left(\frac{a}{b}\right)^2+\left(\frac{b}{a}\right)^2-1\right]=2\sqrt{n}$$

$$\frac{a^2+b^2}{ab}\left[\frac{a^4-b^4}{a^2b^2}-1\right]=2\sqrt{n}$$

$$\frac{a^2+b^2}{ab}\left[\frac{a^4+b^4-a^2b^2}{a^2b^2}\right]=2\sqrt{n}$$

$$\frac{(a^2+b^2)}{a^3b^3}\left[a^4+b^4-a^2b^2\right]=2\sqrt{n} \qquad \text{...........(iii)}$$

From equation (ii)

$$\frac{a^2+b^2}{a^3b^3}\left[-a^2b^2-a^2b^2\right]=2\sqrt{n}$$

$$\frac{a^2+b^2}{a^3b^3}\left[-2a^2b^2\right]=2\sqrt{n}$$

$$-2\left(\frac{a^2+b^2}{ab}\right)=2\sqrt{n}$$

from (i)

$$\frac{-(ab)}{ab}=\sqrt{n}$$

$\sqrt{n}$ = 1

n = 1